Random walks and physical models on infinite graphs: an introduction

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Abstract. This paper is a review of some basic mathematical ideas and results, concerning the relations between random walks and physical models on infinite graphs from the physicists point of view. The presentation is mainly focused on statistical models, which are particularly relevant in the physics of matter and in field theory.

Contents

1 Introduction

In the last twenty years, theoretical physicists have shown an increasing interest in random walks on infinite graphs, connected with the study of physical properties of inhomogeneous and disordered systems in the thermodynamic limit.

A relevant number of papers appeared on this subject, concerning applications to polymers, glasses, fractals, amorphous solids, disordered magnets, biological matter, electronic states, diffusion and transport phenomena (e.g., see [1, 5, 21, 24, 25, 28, 30, 32]). In the meanwhile, a specific mathematical formalism has been introduced in the physical literature to deal with such kind of problems, and a new language has developed among the researchers involved in this field, often alternative to the usual graph-theoretical one. Moreover, the study of physical problems on infinite graphs led to definition of brand new mathematical concepts and to the proof of theorems concerning them.

Only recently a real collaboration between mathematicians and physicists working on models on infinite graphs has begun, due to the initiative of both (Statistical mechanics and graph theory 2000 ICTP Trieste, Random Walks and Statistical Physics 2001 ESI Wien). To improve the exchange of ideas and expertise of the two communities, a common effort of "translation" of basic concepts and tools of each field is now of primary importance. This paper has to be viewed as a first relevant step in this direction from the physicists-side.

Our main aim is to give a self contained introduction to some basic physical models on infinite graphs, emphasizing several mathematical details, usually skipped in the papers written by physicists. Therefore, we decided to limit ourselves to a restricted class of fundamental ideas and results, which can be rigorously stated and proven. Due to this choice, many interesting topics are not discussed here, such as electrical networks, magnetic models and quantum models; for all of them, we refer the reader to the existing literature for more specific applications.

One of the most difficult problems in our task is undoubtedly the "physical reality" hypothesis implicit in all physical works: by this term we mean a series of unexpressed conditions sufficient to produce a set of behaviours observed in real systems.

Let us give an example: all real physical structures (embedded in three-dimensional space) have been found up to now to exhibit power law behaviour in the low-frequency density of vibrational states and therefore, when considering an infinite graph where a physical model is defined, one always assumes that it satisfies the (often unknown) mathematical conditions sufficient to produce such a behaviour. In our opinion the study of these "physical reality" conditions is now the most promising and interesting field for a fruitful collaboration between mathematicians and physicists. To this aim, we always explicitly state all the mathematical conditions usually assumed in physical literature, pointing out in the Remarks the still open points or only heuristically "solved" problems.

We hope that this review will be useful to the mathematical community from at least two different point of view: first, it would offer a collection of unsolved mathematical problems, whose solution would be of great importance to physics; second, it should make the interested reader able to understand the language and the ideas which can be found in advanced physical literature concerning infinite graphs.

1.1 Definitions and notations

Let us introduce some definitions and notations that will be useful in the rest of the paper [4, 20, 23].

Definition 1.1. A graph X is a countable set V_X of vertices (or sites) (i) connected pairwise by a set E_X of unoriented edges (or links) $(i, j) = (j, i)$. Two connected vertices are called nearest neighbours. We denote by z_i the connectivity of the site i, i.e. the number of its nearest neighbours.

Definition 1.2. A path in X is a sequence of consecutive edges $\{(i, k)(k, h) \dots (n, m)(m, j)\}$ and its length is the number of edges in the sequence. A graph is said to be connected if, for any two vertices $i, j \in V_X$, there is always a path joining them.

Definition 1.3. The adjacency matrix A_{ij} is:

$$A_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E_X, \\ 0 & \text{if } (i, j) \notin E_X. \end{cases} \tag{1.1}$$

Definition 1.4. The Laplacian matrix Δ_{ij} is:

$$\Delta_{ij} = z_i \delta_{ij} - A_{ij}. \tag{1.2}$$

Notice that: $z_i = \sum_j A_{ij}$. We define $Z_{ij} = z_i \delta_{ij}$.

A generalization of the Laplacian matrix can be given:

Definition 1.5. The matrix J_{ij} is called a ferromagnetic coupling matrix, if $\exists J_{\max}$, $J_{\min} \in \mathbb{R}^+$:

max, min –> max, min, throughout

$$J_{ji} = J_{ij} = \begin{cases} J_{\min} < J_{ij} < J_{\max} & \text{if } (i, j) \in E_X, \\ 0 & \text{if } (i, j) \notin E_X. \end{cases} \tag{1.3}$$

The generalized Laplacian associated to J_{ij} is:

$$L_{ij} = I_i \delta_{ij} - J_{ij}. \tag{1.4}$$

where $I_i = \sum_j J_{ij}$. We also define $I_{ij} = I_i \delta_{ij}$.

2 The thermodynamic limit

2.1 Distance, Van Hove sphere and growth exponent

Unoriented graphs are naturally provided with an intrinsic distance, which in physics is called the chemical distance $r_{i,j}$.

Definition 2.1. $r_{i,j}$ is the length of the shortest path connecting the vertices i and j. The distance between i and the subset $V' \subset V_X$ is $d(i, V') = \inf\{r_{i,k} \in \mathbb{N} | \, k \in V'\}$.

The chemical distance defines on the graph the balls of radius $r \in \mathbb{N}$ and center $o \in V_X$. In the physical literature these subgraphs are called the Van Hove spheres $\mathcal{S}_{o,r}$.

Definition 2.2. $\mathcal{S}_{o,r}$ is the subgraph of X, determined by the set of vertices $V_{o,r} = \{i \in V | r_{i,o} \leq r\}$ and by the set of edges $E_{o,r} = \{(i, j) \in E | i \in V_{o,r}, \, j \in V_{o,r}\}$. The border of $\mathcal{S}_{o,r}$ is given by the set $\partial V_{o,r} = \{i \in V_{o,r} | \exists j \in V_X, (i, j) \in E_{o,r}, \, j \notin V_{o,r}\}$. For $V' \subset V_X$, we also define $\tilde{V}_{V',r} = \{i \in V_X | \, d(i, V') \leq r\}$.

In some cases it is useful to introduce sequences of generalized spheres $\mathcal{S}'_{o,r}$, defined by sets $V'_{o,r} \subset V_X$ such that $V'_{o,0} = \{o\}$, $V'_{o,r} \subset V'_{o,r+1}$ and $\bigcup_{r=0}^{\infty} V'_{o,r} = V_X$ and by the sets of edges $E'_{o,r} = \{(i, j) \in E | i \in V'_{o,r}, \, j \in V'_{o,r}\}$. Here we always use for the sphere Definition 2.2.

Let $|S|$ be the cardinality of a set S. Then $|V_{o,r}|$, as a function of the distance r, describes the growth rate of the graph at the large scale [26]. In particular:

Definition 2.3. A graph is said to have a polynomial growth if $\forall o \in V_X \; \exists c, k$, such that $|V_{o,r}| < c \, r^k$.

Definition 2.4. For a graph satisfying (2.3), we define the upper growth exponent d_g^+ and the lower growth exponent d_g^- as $d_g^+ = \inf\{k| \, |V_{o,r}| < c_1 \, r^k, \forall o \in V\}$ and $d_g^- = \sup\{k| \, |V_{o,r}| > c_2 \, r^k, \forall o \in V\}$. If $d_g^+ = d_g^-$ we call them the growth exponent d_g, or the classical connectivity dimension.

The connectivity dimension d_g is known for a large class of graphs: on lattices $\mathbb{Z}^d$ it coincides with the usual Euclidean dimension d, and for many fractals it has been exactly evaluated [21].

2.2 Physical conditions

Discrete structures describing real physical systems are characterized by some important properties, which can be translated in mathematical requirements for the graphs we will consider.

p.c.1 We will consider only connected graphs (Definition 1.2), since any physical model on disconnected structures can be reduced to the separate study of the models defined on each connected component and hence to the case of connected graphs.

p.c.2 Since physical interactions are always bounded, the coordination numbers z_i, representing the number of neighbours interacting with the site i, have to be bounded; i.e. $\exists z_{\max} \mid z_i \leq z_{\max} \; \forall i \in V_X$.

p.c.3 Real systems are always embedded in 3-dimensional space. This constraint requires for the graph $\mathcal{G}$ the conditions:

(a) X has a polynomial growth (Definition 2.3)

(b)

$$\lim_{r\to\infty} \frac{|\partial V_{o,r}|}{|V_{o,r}|} = 0 \tag{2.5}$$

The existence itself of the limit is a physical requirement on $\mathcal{G}$.

Some interesting graphs such as the Bethe lattice do not satisfy (a) and (b). For this kind of structures many results we give in this paper do not apply and one has to introduce different techniques.

Remark 2.5. For a large class of physically interesting graphs we have considered so far, conditions (a) and (b) appears to be equivalent. However for the equivalence of the two conditions a rigorous result is still lacking.

A graph satisfying **p.c.1**, **p.c.2** and **p.c.3** will be called *physical graph* $\mathcal{G}$ and the sets of its vertices and edges will be denoted respectively with V an E. **p.c.1** and **p.c.2** represent strong constraints on $\mathcal{G}$ and, as we will prove in detail, they have very important consequences. For example, **p.c.1** implies a simple but important limitation on the difference of size for spheres of different centers.

Theorem 2.6. *Given a physical graph $\mathcal{G}$, let $\mathcal{S}_{o,r}$ and $\mathcal{S}_{o',r}$ be two spheres of centers o and o', respectively, and radius r. One has:*

$$||V_{o,r}| - |V_{o',r}|| \leq (z_{\max})^{2r_{o,o'}} |\partial V_{o,r}|. \tag{2.6}$$

Proof. Since $V_{o',r} \subset V_{o,r+r_{o,o'}}$,

$$|V_{o',r}| \leq |V_{o,r+r_{o,o'}}| \leq |V_{o,r}| + |V_{o,r+r_{o,o'}} \triangle V_{o,r}|,$$

where $\triangle$ denotes the symmetric difference. Now we have $|V_{o,r} \triangle V_{o,r+r_{o,o'}}| < |\widetilde{V}_{\partial V_{o,r}, r_{o,o'}}|$, where $|\widetilde{V}_{\partial V_{o,r}, r_{o,o'}}|$, as in Definition 2.2, is the number of sites whose distance from $\partial V_{o,r}$ is smaller than $r_{o,o'}$. Form the uniform boundedness of z_i one obtains $|\widetilde{V}_{\partial V_{o,r}, r_{o,o'}}| \leq (z_{\max})^{r_{o,o'}} |\partial V_{o,r}|$, and then:

$$|V_{o',r}| \leq |V_{o,r}| + (z_{\max})^{r_{o,o'}} |\partial V_{o,r}|. \tag{2.7}$$

From the properties of the distance, $r_{i,o} - r_{o,o'} \leq r_{i,o'} \leq r_{i,o} + r_{o,o'}$, hence $\forall i$ such that $r_{i,o} = r$ (i.e $i \in \partial V_{o,r}$) we have $r - r_{o,o'} \leq r_{i,o'} \leq r + r_{o,o'}$ and $i \in \widetilde{V}_{\partial V_{o,r}, r_{o,o'}}$. So again from boundedness of z_i:

$$|\partial V_{o,r}| \leq (z_{\max})^{r_{o,o'}} |\partial V_{o',r}|. \tag{2.8}$$

Inequality (2.6) is a simple consequence of (2.7) and (2.8). □

2.3 Averages in the thermodynamic limit and sets measure

Thermodynamic averages have a crucial role in the study of statistical models on discrete structures. This requires the introduction of infinite graphs and the study of the limit $r \to \infty$ for the Van Hove spheres [13].

Definition 2.7. Given a physical graph $\mathcal{G}$, let $\phi_i : V \to \mathbb{R}$. The average in the thermodynamic limit of ϕ_i is:

$$\overline{\phi} \equiv \lim_{r\to\infty} \frac{\sum_{i\in V_{o,r}} \phi_i}{|V_{o,r}|}. \tag{2.9}$$

The existence itself of limit (2.9) is a physical requirement on the functions ϕ_i.

In [2] more general averages are defined giving to each site a weight $\lambda_{i,r}$. Physical constrains on graph structures, given in Section 2.2, has important consequences for the behaviour of the thermodynamic averages, such as the independence of the limit (2.9) from the choice of the center o.

Theorem 2.8. *Let $\mathcal{G}$ be a physical graph and $\phi_i : V \to \mathbb{R}$ a function bounded from below, i.e. $\phi_i > \phi_{\min}$ $\forall i \in V$. If limit* (2.9) *exists for the Van Hove spheres of center o', then it exists for any possible center o and the result does not depend on o.*

Proof. For any two vertices o and o', we have:

$$\begin{aligned} &\frac{\sum_{i\in V_{o',r-r_{o,o'}}} \phi_i \;+\; \sum_{i\in V_{o,r}\triangle V_{o',r-r_{o,o'}}} \phi_i}{|V_{o,r}|} \\ &\quad= \frac{\sum_{i\in V_{o,r}} \phi_i}{|V_{o,r}|} = \frac{\sum_{i\in V_{o',r+r_{o,o'}}} \phi_i \;-\; \sum_{i\in V_{o',r+r_{o,o'}}\triangle V_{o,r}} \phi_i}{|V_{o,r}|}, \end{aligned} \tag{2.10}$$

where $V_{o,r} \subseteq V_{o',r+r_{o,o'}}$, $V_{o',r-r_{o,o'}} \subseteq V_{o,r}$. From the boundedness of ϕ_i:

$$\begin{aligned} &\frac{\sum_{i\in V_{o',r-r_{o,o'}}} \phi_i - \phi_{\min}|V_{o,r} \triangle V_{o',r-r_{o,o'}}|}{|V_{o,r}|} \\ &\quad\leq \frac{\displaystyle\sum_{i\in V_{o,r}} \phi_i}{|V_{o,r}|} \leq \frac{\sum_{i\in V_{o',r+r_{o,o'}}} \phi_i + \phi_{\min}|V_{o',r+r_{o,o'}} \triangle V_{o,r}|}{|V_{o,r}|}. \end{aligned} \tag{2.11}$$

In analogy with (2.6) one proves:

$$\begin{aligned}
|V_{o,r} \triangle V_{o',r-r_{o,o'}}| &\le (z_{\max})^{r_{o,o'}} |\partial V_{o,r}|, \\
|V_{o,r} \triangle V_{o',r-r_{o,o'}}| &\le (z_{\max})^{r_{o,o'}} |\partial V_{o,r-r_{o,o'}}|, \\
|V_{o,r+r_{o,o'}} \triangle V_{o',r}| &\le (z_{\max})^{r_{o,o'}} |\partial V_{o,r+r_{o,o'}}|, \\
|V_{o,r+r_{o,o'}} \triangle V_{o',r}| &\le (z_{\max})^{r_{o,o'}} |\partial V_{o,r}|,
\end{aligned} \tag{2.12}$$

and with property (2.5) we get:

$$\lim_{r\to\infty} \frac{\sum_{i\in V_{o',r-r_{o,o'}}} \phi_i}{|V_{o,r}|} \le \lim_{r\to\infty} \frac{\sum_{i\in V_{o,r}} \phi_i}{|V_{o,r}|} \le \lim_{r\to\infty} \frac{\sum_{i\in V_{o',r+r_{o,o'}}} \phi_i}{|V_{o,r}|},$$

and

$$\begin{aligned}
&\lim_{r\to\infty} \frac{\sum_{i\in V_{o',r-r_{o,o'}}} \phi_i}{|V_{o',r-r_{o,o'}}| + |V_{o,r} \triangle V_{o',r-r_{o,o'}}|} \\
&\quad \le \lim_{r\to\infty} \frac{\sum_{i\in V_{o,r}} \phi_i}{|V_{o,r}|} \le \lim_{r\to\infty} \frac{\sum_{i\in V_{o',r+r_{o,o'}}} \phi_i}{|V_{o',r+r_{o,o'}}| - |V_{o,r} \triangle V_{o',r-r_{o,o'}}|}.
\end{aligned} \tag{2.13}$$

Using again property (2.5) and inequalities (2.12) we get:

$$\lim_{r\to\infty} \frac{\sum_{i\in V_{o',r-r_{o,o'}}} \phi_i}{|V_{o',r-r_{o,o'}}|} \le \lim_{r\to\infty} \frac{\sum_{i\in V_{o,r}} \phi_i}{|V_{o,r}|} \le \lim_{r\to\infty} \frac{\sum_{i\in V_{o',r+r_{o,o'}}} \phi_i}{|V_{o',r+r_{o,o'}}|}. \tag{2.14}$$

Therefore, if the limit with the spheres centered in o' exists, it gives the same result using as center any vertex o. □

In what follows we drop the index o when we evaluate thermodynamic averages. Now we can define the measure of the subsets of $V' \subset V$.

Definition 2.9. Given a physical graph $\mathcal{G}$, the measure of a subset $V' \subset V$ is $\|V'\| = \overline{\chi(V')}$, where $\chi_i(V')$ is the characteristic function defined as $\chi_i(V') = 1$ if $i \in V'$ and $\chi_i(V') = 0$ if $i \notin V'$. The measure of a subset of edges $E' \subset E$ is $\lim_{r\to\infty} |E'_r|/|V_r|$, where $E'_r = \{(i, j) \in E' |\ i \in V_r,\ j \in V_r\}$.

Since $\chi_i(V')$ is bounded from below, when the thermodynamic average exists, the value of the measure $\|V'\|$ does not depend on the choice of the center o. Unfortunately in some cases the limit defining the measure does not exist. A typical example is the subset of $\mathbb{Z}$ defined as $\{i \in \mathbb{Z}|\ 2^{2n} \le |i| \le 2^{2n+1}, \forall\ n \in \mathbb{N}\}$. However, these subsets are not very interesting from a physical point of view, for example, they cannot characterize sites with a certain thermodynamic property, since this property should not be additive. Hence we will consider only subsets with a well-defined measure.

3 Random walks

3.1 Definitions

Let us begin by recalling the basic definitions and results concerning (simple) random walks on infinite graphs. A more detailed treatment can be found in the mathematical reviews by Woess [35, 36].

Definition 3.1. The (simple) random walk on a graph X is defined by the jumping probability p_{ij} between nearest neighbours sites i and j:

$$p_{ij} = \frac{A_{ij}}{z_i} = (Z^{-1}A)_{ij}, \tag{3.15}$$

where $Z_{ij} = z_i \delta_{ij}$. The probability of reaching in t steps site j starting from i is:

$$P_{ij}(t) = (p^t)_{ij}. \tag{3.16}$$

We denote by $F_{ij}(t)$ the probability for a walker starting from i of reaching for the first time in t steps the site $j \neq i$, and by $F_{ii}(t)$ the probability of returning to the starting point i for the first time after t steps ($F_{ii}(0) = 0$).

The basic relationship between $P_{ij}(t)$ and $F_{ij}(t)$ is given by:

$$P_{ij}(t) = \sum_{k=0}^{t} F_{ij}(k) P_{jj}(t-k) + \delta_{ij}\delta_{t0}. \tag{3.17}$$

$F_{ij} \equiv \sum_{t=0}^{\infty} F_{ij}(t)$ turns out to be the probability of ever reaching the site j starting from i (or of ever returning to i if $j = i$). Therefore, $0 < F_{ij} \leq 1$.

Definition 3.2. The generating functions $\widetilde{P}_{ij}(\lambda)$ and $\widetilde{F}_{ij}(\lambda)$ are given by

$$\widetilde{P}_{ij}(\lambda) = \sum_{t=0}^{\infty} \lambda^t P_{ij}(t), \quad \widetilde{F}_{ij}(\lambda) = \sum_{t=0}^{\infty} \lambda^t F_{ij}(t), \tag{3.18}$$

where λ is a complex number.

Abel –> From definition (3.18) by Abel's lemma we have that $\widetilde{F}_{ij}(\lambda)$ and $\widetilde{P}_{ij}(\lambda)$ are $\mathcal{C}^\infty$
Abel's functions in $[0, 1)$. Furthermore, $\widetilde{F}_{ij}(\lambda)$ is continuous also for $\lambda = 1$, while $\widetilde{P}_{ij}(\lambda)$ can diverge at this point.

Multiplying equations (3.17) by λ^t and then summing over all possible t we get:

$$\widetilde{P}_{ij}(\lambda) = \widetilde{F}_{ij}(\lambda)\widetilde{P}_{jj}(\lambda) + \delta_{ij}. \tag{3.19}$$

Lemma 3.3. *A simple bound on $\widetilde{F}_{ij}(\lambda)$ and on $\widetilde{P}_{ij}(\lambda)$ is given by:*

$$\widetilde{P}_{ij}(\lambda) \leq (1-\lambda)^{-1}, \quad \widetilde{F}_{ij}(\lambda) \leq (1-\lambda)^{-1}. \tag{3.20}$$

Proof. From $P_{ij}(t) < 1$, $F_{ij}(t) < 1$ and (3.18) one immediately obtains (3.20). □

In the following we will use the notations $\widetilde{P}_i(\lambda) \equiv \widetilde{P}_{ii}(\lambda)$ and $\widetilde{F}_i(\lambda) \equiv \widetilde{F}_{ii}(\lambda)$.

3.2 The local type problem

Infinite graphs can be classified by the long time asymptotic behaviour of simple random walks and in particular by the quantities $\widetilde{F}_i(1)$ and $\lim_{\lambda\to 1} \widetilde{P}_i(\lambda)$ [29].

Definition 3.4. A graph X is called *locally recurrent* if

$$\widetilde{F}_i(1) = 1 \quad \text{or, equivalently,} \quad \lim_{\lambda\to 1} \widetilde{P}_i(\lambda) = \infty \quad \forall i \in V_X. \tag{3.21}$$

On the other hand, X is called *locally transient* if:

$$\widetilde{F}_i(1) < 1 \quad \text{or, equivalently,} \quad \lim_{\lambda\to 1} \widetilde{P}_i(\lambda) < \infty \quad \forall i \in V_X. \tag{3.22}$$

The equivalences in the definitions (3.21) and (3.22) are simple consequences of equation (3.19). By using standard properties of Markov chains one can prove that (3.21) and (3.22) are independent of the vertex i [35], and then Definition 3.4 can be considered as a property of the graph itself.

Local transience and local recurrence satisfy important universality properties [35]. Indeed, local transience and recurrence do not change if we replace the jumping probabilities of the random walk (3.15) with the generalized jumping probabilities:

$$p_{ij} = \frac{J_{ij}}{I_i}. \tag{3.23}$$

In [35] the invariance of the local recurrence properties under a wide class of transformations of the graph itself is also proven. Local recurrence and transience are not modified by the addition a finite number of links or the introduction of second neighbour links on the graph. These invariances put into evidence that local recurrence and transience are determined only by the large scale topology of the graph.

3.3 The local spectral dimension

The behaviour of $\widetilde{P}_i(\lambda)$ for $\lambda \to 1^-$ can be used not only to classify the graph as locally transient or recurrent, but also to introduce the local spectral dimension $\widetilde{d}$ which can be considered as a finer invariant of the graph topology. The spectral dimension has been widely studied in physics [1, 5, 32], since it is closely connected with such important phenomena as the anomalous diffusion and the vibrational spectra of harmonic oscillations. In the following we will use the definition given in [24]. Since $\widetilde{P}_i(\lambda)$ for $\lambda < 1$ is a $\mathcal{C}^\infty$ differentiable function one can define the degree of recurrence of a graph.

Definition 3.5. Let $\widetilde{P}_i^{(n)}(\lambda)$ be:

$$\widetilde{P}_i^{(n)}(\lambda) = \left(\frac{d}{d\lambda}\right)^n \widetilde{P}_i(\lambda). \tag{3.24}$$

A graph $\mathcal{G}$ is recurrent of degree N if:

$$\lim_{\lambda\to 1^-} \widetilde{P}_i^{(n)}(\lambda) < \infty,\ \forall n < N, \quad \text{and} \quad \lim_{\lambda\to 1^-} \widetilde{P}_i^{(N)}(\lambda) = \infty. \tag{3.25}$$

Definition 3.6. Let X be recurrent of degree N. If the limit

$$D = \lim_{\lambda\to 1^-} \frac{\log(\widetilde{P}_i^{(N)}(\lambda))}{-\log(1-\lambda)} \tag{3.26}$$

exists, then the spectral dimension is $\widetilde{d} = 2(N - D + 1)$.

Lemma 3.7. *Let X be a recurrent of degree N graph with local spectral dimension $\widetilde{d}$. We have $\widetilde{d} \le 2$ if $N = 0$ and $2N \le \widetilde{d} \le 2(N+1)$ for $N \ge 1$.*

Proof. Since $\widetilde{P}_i^{(N)}(\lambda)) > 1$, we have $D \ \ge 0$ and the case $N = 0$ is proven. For the case $N \ge 1$ we have to show that $D \le 1$. Let us suppose that $D > 1$. We will prove that $\lim_{\lambda\to 1^-} \widetilde{P}_i^{(N-1)}(\lambda) = \infty$, hence $\mathcal{G}$ is recurrent of degree $N-1$ leading to a contradiction. From (3.26) we have that $\forall\epsilon > 0$, $\exists\lambda'$ such that $\forall\lambda''$, $\lambda' < \lambda'' < 1$

$$\widetilde{P}_i^{(N)}(\lambda'') > (1-\lambda'')^{-(D-\epsilon)}. \tag{3.27}$$

Integrating (3.27) between λ' and λ, we get:

$$\begin{aligned}\widetilde{P}_i^{(N-1)}(\lambda) > (D-1-\epsilon)^{-1}\big((1-\lambda)^{-(D-1-\epsilon)} - (1-\lambda')^{-(D-1-\epsilon)}\big)\\ + \widetilde{P}_i^{(N-1)}(\lambda').\end{aligned} \tag{3.28}$$

Hence, $\lim_{\lambda\to 1^-} \widetilde{P}_i^{(N-1)}(\lambda) = \infty$. □

In [24] the independence of $\widetilde{d}$ of the choice of site i is proven. Therefore, the local spectral dimension can be considered as a property of the graph. Furthermore, in [24] some invariance properties such as the invariance for the rescaling of the jumping probability (3.23) are also proven.

Locally recurrent graphs are recurrent of degree 0 and have local spectral dimension smaller than 2. On the Euclidean lattices $\mathbb{Z}^d$, $\widetilde{d} = d$ [27], hence $\widetilde{d}$ can be considered as a generalization of the usual notion of dimension for lattices. Moreover, $\widetilde{d}$ has been evaluated for many graphs such as exactly decimable fractals [24, 31] and bundled structures [3, 17, 18, 34] (The Sierpinski gasket in Fig. 1 with $\widetilde{d} = 2\log(3)/\log(5)$ and the comb graph in Fig. 2 with $\widetilde{d} = 1.5$ are two typical examples of exactly decimable fractals and bundled structures).

Definition 3.5 is more general than the usual definition of the local spectral dimension given in physics, i.e. $\widetilde{P}_i(\lambda) \sim (1-\lambda)^{\widetilde{d}/2-1}$, ($\sim$ denotes the singular asymptotic behaviour). A typical example is the Sierpinski gasket (Fig. 1), which has been widely studied in physics [31]. From Definition 3.5, this structure has dimension $\widetilde{d} = 2\log(3)/\log(5)$. However in [22] it is proven that the asymptotic behaviour of $\widetilde{P}_i(\lambda)$ is more complex since it presents also a small oscillatory part (here $a \ll 1$):

$$\widetilde{P}_i(\lambda) \sim (1-\lambda)^{\widetilde{d}/2-1-N}(1 - a\sin(b\log((1-\lambda)) + c)).$$

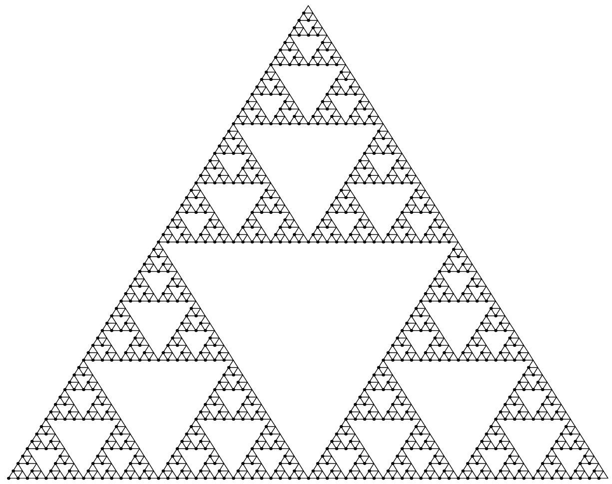

Figure 1. The Sierpinski gasket

Figure 2. The comb graph

Remark 3.8. The existence of the local spectral dimension for any graph recurrent of degree N is an important mathematically open point. Indeed, all the known graphs, for which $\widetilde{d}$ can not be defined (an example is the inhomogeneous Bethe lattice of Fig. 3), are not recurrent of any degree. Moreover, in all the examples we have studied up to now, the local spectral dimension is well-defined for all physical graphs $\mathcal{G}$. Proving that these are indeed sufficient conditions for the existence of $\widetilde{d}$ is another interesting open mathematical problem.

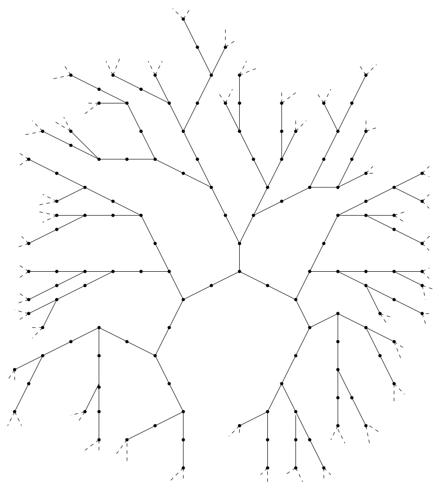

Figure 3. The inhomogeneous Bethe lattice

4 Thermodynamic averages and random walks

4.1 Recurrence and transience on the average

The study of thermodynamic properties of statistical models on infinite graphs requires the introduction of averages of local quantities. The latter are related to random walks by the return probabilities on the average $\overline{P}$ and $\overline{F}$ [13].

Definition 4.1. Given a physical graph $\mathcal{G}$, the return probabilities on the average $\overline{P}$ and $\overline{F}$ are defined by:

$$\overline{P} = \lim_{\lambda \to 1} \overline{\widetilde{P}(\lambda)} \tag{4.29}$$

$$\overline{F} = \lim_{\lambda \to 1} \overline{\widetilde{F}(\lambda)} \tag{4.30}$$

Definition 4.2. $\mathcal{G}$ is called *recurrent on the average* (ROA) if $\overline{F} = 1$, while it is *transient on the average* (TOA) when $\overline{F} < 1$.

Remark 4.3. The main mathematical point in Definitions 4.1 and 4.2 is the existence of the thermodynamic average for the functions $\widetilde{F}_i(\lambda)$ and $\widetilde{P}_i(\lambda)$. The existence of this limit will always be assumed for physical graphs. In [2] an example when $\overline{\widetilde{P}(\lambda)}$ and $\overline{\widetilde{F}(\lambda)}$ are not well-defined is presented. However, the graph of this example does not satisfy **p.c.3**. On the other hand, $\overline{\widetilde{P}(\lambda)}$ and $\overline{\widetilde{F}(\lambda)}$ are well-defined for all physical graphs $\mathcal{G}$ we have studied up to now. A general result in this direction would be an important breakthrough in understanding the average properties of random walks on graphs.

Under the hypothesis of the existence of the thermodynamic averages, the limit $\lambda \to 1^-$ is always well-defined since $\overline{\widetilde{P}(\lambda)}$ and $\overline{\widetilde{F}(\lambda)}$ are increasing functions of λ. Furthermore, the independence of the averages of the center of the spheres is assured by Theorem 2.8 and by the boundedness from below of $\widetilde{F}_i(\lambda)$ and $\widetilde{P}_i(\lambda)$. Hence Definition 4.2 represents a property of the graph.

In [2] a different definition of transience and recurrence on the average is given. There the thermodynamic limit $\lim_{r\to\infty}$ is replaced with $\liminf_{r\to\infty}$ which is always well-defined. Moreover, in [2] the limit $\lambda \to 1^-$ is evaluated before taking the thermodynamic average. This definition leads to another graph classification. For example, the chain of increasing cubes (see Fig. 4) from Definition 4.2 is a TOA graph, whereas it is recurrent on the average according to the definition from [2]. Furthermore, the condition for the limit to be independent of the center of the sphere in this case is weaker than the hypotheses of Section 2.2.

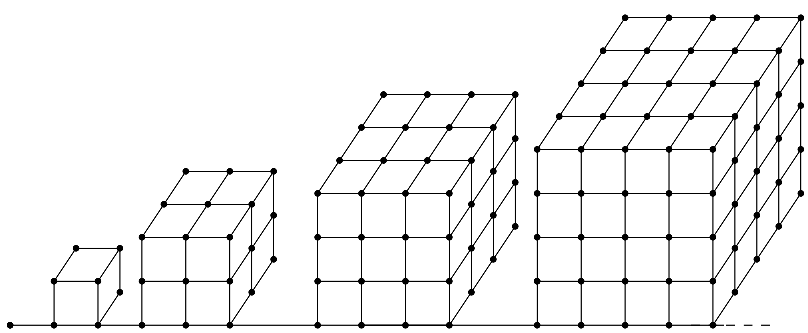

Figure 4. The chain of increasing cubes

Recurrence and transience on the average are in general independent from the corresponding local properties. The first example of this phenomenon occurring on inhomogeneous structures was found in a class of infinite trees called NTD

(Fig. 5) which are locally transient but recurrent on the average [10]. On the other hand, the chain of increasing cubes in Fig. 4 is an example of locally recurrent but transient on the average graph.

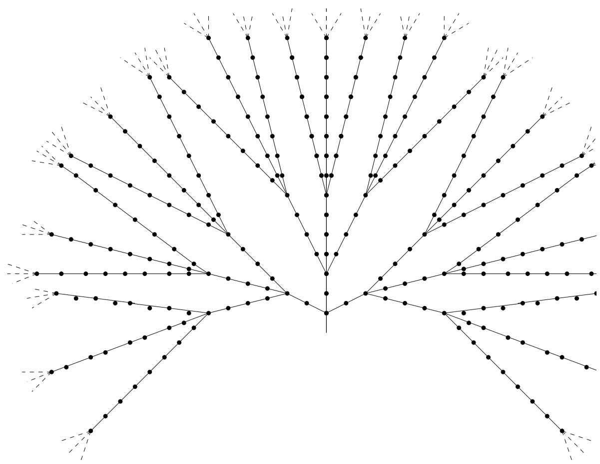

Figure 5. The NTD graph

For (4.30) and (4.29) we cannot prove any simple relation between (4.30) and (4.29) analogous to equation (3.19) for local probabilities. Indeed, averaging (3.19) over all sites i would involve the average of a product, which, due to correlations, is in general different from the product of the averages. Therefore, the equivalence $\widetilde{F_i}(1) = 1 \Leftrightarrow \lim_{\lambda\to 1} \widetilde{P_i}(\lambda) = \infty$ is not true. There are graphs for which $\overline{F} < 1$, but $\overline{P} = \infty$ (an example is shown in Fig. 6), and the study of the relation between $\overline{P}$ and $\overline{F}$ is a non-trivial problem, which will be dealt with in detail in Sections 4.4 and 4.3.

Lemma 4.4. *Let $\mathcal{G}$ be a physical graph such that*

$$\overline{P}(t) = \lim_{r\to\infty} |V_r|^{-1} \sum_{i\in V_r} P_{ii}(t) \quad \text{and} \quad \overline{F}(t) = \lim_{r\to\infty} |V_r|^{-1} \sum_{i\in V_r} F_{ii}(t)$$

are well-defined. For all $\lambda < 1$ we have:

$$\overline{\widetilde{F}(\lambda)} = \sum_{t=0}^{\infty} \lambda^t \overline{F(t)}, \quad \overline{\widetilde{P}(\lambda)} = \sum_{t=0}^{\infty} \lambda^t \overline{P(t)}. \tag{4.31}$$

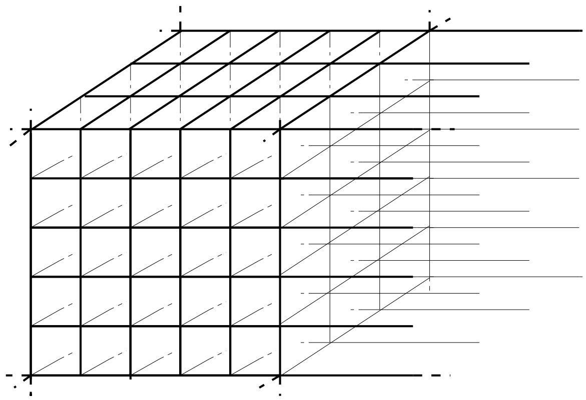

Figure 6. An example of mixed TOA graph

Proof. For all $\lambda < 1$

$$\begin{aligned}\overline{\widetilde{P}(\lambda)} &= \lim_{r\to\infty} \sum_{i\in V_r} |V_r|^{-1} \Big(\sum_{t=0}^{\bar{t}} \lambda^t P_{ii}(t) + \sum_{t=\bar{t}}^{\infty} \lambda^{\bar{t}} P_{ii}(t) \Big) \\ &= \sum_{t=0}^{\bar{t}} \lambda^t \overline{P(t)} + \lim_{r\to\infty} \sum_{i\in V_r} |V_r|^{-1} \sum_{t=\bar{t}}^{\infty} \lambda^t P_{ii}(t).\end{aligned} \tag{4.32}$$

Since $\sum_{i\in S_{o,r}} \|V_r\|^{-1} \sum_{t=\bar{t}}^{\infty} \lambda^{\bar{t}} P_{ii}^{\mathcal{G}'}(t) \le \lambda^{\bar{t}}(1-\lambda)^{-1}$, letting in (4.32) $\bar{t} \to \infty$ we get (4.31). An analogous equation also holds for $F_{ii}(t)$. □

Remark 4.5. In the following we will also assume that $\overline{P(t)}$ and $\overline{F(t)}$ are well-defined on a physical graph. Finding general conditions under which these hypotheses hold is another important mathematical open point in the study of random walks on the average.

Lemma 4.4 shows that the series defining the generating functions and the thermodynamic averages commute. Moreover, from (4.31) and from the Abel lemma we get that $\overline{\widetilde{P}(\lambda)}$ and $\overline{\widetilde{F}(\lambda)}$ are $\mathcal{C}^{\infty}$ functions in $[0, 1)$. The function $\widetilde{F}_{ij}(\lambda)$ is continuous also at $\lambda = 1$, while $\widetilde{P}_{ij}(\lambda)$ can diverge at this point.

4.2 The average spectral dimension

Since $\overline{\widetilde{P}(\lambda)}$ is $\mathcal{C}^\infty$ in [0, 1), one can define the average spectral dimension $\overline{d}$ in a way analogous to $\widetilde{d}$. Interestingly, the average spectral dimension of real inhomogeneous discrete structures can be experimentally measured [28]. Moreover, $\overline{d}$ has a great influence on the behaviour of the thermodynamic quantities (such as the specific heat) of physical models. Hence $\overline{d}$ is a fundamental quantity in statistical and condensed matter physics.

Definition 4.6. Let $\overline{\widetilde{P}(\lambda)}^{(n)}$ be

$$\overline{\widetilde{P}_i(\lambda)}^{(n)} = \left(\frac{d}{d\lambda}\right)^n \overline{\widetilde{P}(\lambda)}. \tag{4.33}$$

A physical graph $\mathcal{G}$ is recurrent on the average of degree N if

$$\lim_{\lambda\to 1^-} \overline{\widetilde{P}(\lambda)}^{(n)} < \infty,\ \forall n < N \quad \text{and} \quad \lim_{\lambda\to 1^-} \overline{\widetilde{P}(\lambda)}^{(N)} = \infty. \tag{4.34}$$

Definition 4.7. Let $\mathcal{G}$ be recurrent on the average of degree N, and

$$D = \lim_{\lambda\to 1^-} \frac{\log\left(\overline{\widetilde{P}(\lambda)}^{(n)}\right)}{-\log(1-\lambda)}. \tag{4.35}$$

If the limit (3.26) exists, the average spectral dimension is $\overline{d} = 2(N - D + 1)$.

Lemma 4.8. *Let $\mathcal{G}$ be a recurrent of degree N graph with local spectral dimension $\widetilde{d}$. Then $\widetilde{d} \le 2$ if $N = 0$ and $2N \le \widetilde{d} \le 2(N+1)$ for $N \ge 1$.*

Proof. The proof is completely analogous to that of Lemma 3.7. □

The average spectral dimension has been evaluated for many discrete structures showing that in general, on inhomogeneous graphs, it is different from the local one. We call this phenomenon dynamical dimensional splitting. For example, on the comb graph (Fig. 2) $\widetilde{d} = 1.5$, $\overline{d} = 1$ [19], and on the NTD graph (Fig. 5) $\widetilde{d} = 1 + \log(3)/\log(2)$, $\overline{d} = 1$ [10]. On the other hand, on homogeneous structures, such as the $\mathbb{Z}^d$ lattices, for which all sites are equivalent, we have $\overline{\widetilde{P}(\lambda)} = \widetilde{P}_i(\lambda)$, $\forall i$ and then $\widetilde{d} = \overline{d}$.

Remark 4.9. The behaviour of average quantities $\overline{\widetilde{P}(\lambda)}$ seems to be much more regular than $\widetilde{P}_i(\lambda)$. For example, numerical results for the Sierpinski gasket put into evidence that the oscillations of $\widetilde{P}_i(\lambda)$ [22], which have been described in Section 3.3, disappear in $\overline{\widetilde{P}(\lambda)}$. This is another heuristic result, which requires a rigorous formulation.

Remark 4.10. Even for the average spectral dimension the main open problem from a mathematical point of view is finding general conditions for its existence. As in the case of $\widetilde{d}$, all the known graphs, for which $\overline{d}$ can not be defined, are not even recurrent on the average of any degree. Furthermore, all these graphs do not satisfy **p.c.2** and **p.c.3**.

4.3 Pure and mixed transience on the average

In this section we study the relation between $\overline{P}$ and $\overline{F}$. This problem, as stated in Section 4.1 is not simple as for the case of local recurrence. In particular we show that a complete picture of the behavior of random walks on graphs can be given by dividing transient on the average graphs into two further classes, which will be called *pure* and *mixed* transient on the average (TOA) [13].

Theorem 4.11. *Let $\mathcal{G}$ be an ROA graph* (*i.e.* $\overline{F} = 1$)*, then* $\overline{P} = \infty$.

Proof. Since $\overline{F} = 1$, for each $\delta > 0$ it exists ϵ such that $1 - \epsilon \leq \lambda < 1$. Then we have: $1 - \delta \leq \overline{\widetilde{F}(\lambda)} \leq 1$. Let $S = \{\, i \in V |\ \widetilde{F}_i(1 - \epsilon) < 1 - \sqrt{\delta}\} \subset V$, then

$$\begin{aligned} 1 - \delta \leq \overline{\widetilde{F}(1-\epsilon)} &= \overline{\chi(S)\widetilde{F}(1-\epsilon)} + \overline{\chi(\overline{S})\widetilde{F}(1-\epsilon)} \\ &\leq (1 - \sqrt{\delta})\|S\| + \|\overline{S}\| = 1 - \sqrt{\delta}\|S\| \end{aligned} \tag{4.36}$$

(here $\overline{S}$ denotes the complement of S). From (4.36) we get $\|S\| \leq \sqrt{\delta}$, and then $\|\overline{S}\| \geq 1 - \sqrt{\delta}$. Since $\overline{\widetilde{P}(\lambda)}$ is an increasing function of λ, for each $\lambda \geq 1 - \epsilon$ we get:

$$\overline{\widetilde{P}(\lambda)} \geq \overline{\widetilde{P}(1-\epsilon)} \geq \overline{\chi(\overline{S})(1 - \widetilde{F}(1-\epsilon))^{-1}} \geq \|\overline{S}\|\delta^{-1/2} \geq (1 - \sqrt{\delta})\delta^{-1/2}. \tag{4.37}$$

In this way we have proved that for an arbitrarily large value of $(1-\sqrt{\delta})\delta^{-1/2}$ (as $\delta \to 0$), it exists ϵ such that for each λ with $1-\epsilon \leq \lambda < 1$ we have $\overline{\widetilde{P}(\lambda)} \geq (1-\sqrt{\delta})\delta^{-1/2}$, and therefore $\overline{P} = \lim_{\lambda\to 1} \overline{\widetilde{P}(\lambda)} = \infty$. □

Theorem 4.11 can be easily generalized

Corollary 4.12. *Let $\mathcal{G}$ be a physical graph with a positive measure subset $V' \subseteq V$ such that* $\lim_{\lambda\to 1} \overline{\chi(V')\widetilde{F}(\lambda)} = \|V'\|$. *Then*

$$\overline{P} \geq \lim_{\lambda\to 1} \overline{\chi(S')\widetilde{P}(\lambda)} = \infty \quad \forall S' \subseteq V', \|S'\| > 0. \tag{4.38}$$

Hence we proved that $\overline{F} = 1 \Rightarrow \overline{P} = \infty$. Unfortunately, the inverse relation does not hold (an example is given in Fig. 6), and a further classification is needed.

Definition 4.13. We say that a TOA graph is *mixed* if there exists a subset $V' \subset V$ such that $\|V'\| > 0$ and

$$\lim_{\lambda\to 1} \overline{\chi(V')\widetilde{F}(\lambda)} = \|V'\|. \tag{4.39}$$

Definition 4.14. A graph will be called *pure* TOA, if:

$$\lim_{\lambda\to 1} \overline{\chi(V')\widetilde{F}(\lambda)}\|V'\|^{-1} < k < 1 \quad \forall V' \subseteq V,\ \|V\| > 0. \tag{4.40}$$

Remark 4.15. No TOA graph is known which is neither mixed nor pure. In this case $\lim_{\lambda\to 1} \overline{\chi(V')\widetilde{F}(\lambda)}\|V'\|^{-1}$ should be smaller than 1 $\forall V' \subseteq V$ but not smaller of any real number $k < 1$. We will never consider this case and also for this problem a general result is needed.

Theorem 4.16. *$\overline{P} < \infty$ for any pure TOA graph $\mathcal{G}$.*

Proof. For each $0 < \lambda' < 1$ let $S_{\lambda'} \subseteq V$ be defined as $S_{\lambda'} = \{i \in V|\ \widetilde{F}_i(\lambda') > k\}$. Since $\widetilde{F}_i(\lambda)$ is an increasing function, $\forall\lambda > \lambda'$ we get $\overline{\chi(S_{\lambda'})\widetilde{F}_i(\lambda)} > k\|S_{\lambda'}\|$, and then $\lim_{\lambda\to 1} \overline{\chi(S_{\lambda'})\widetilde{F}_i(\lambda)} > k\|S_{\lambda'}\|$. From Definition 4.14, $\|S_{\lambda'}\| = 0$. Then we get:

$$\begin{aligned}
\overline{\widetilde{P}(\lambda')} &= \overline{\chi(\overline{S}_{\lambda'})\widetilde{P}(\lambda')} + \overline{\chi(S_{\lambda'})\widetilde{P}(\lambda')} \\
&\leq \overline{\chi(\overline{S}_{\lambda'})(1-\widetilde{F}(\lambda'))^{-1}} + \|S_{\lambda'}\|(1-\lambda')^{-1} \\
&\leq \|\overline{S}_{\lambda'}\|(1-k)^{-1} \leq (1-k)^{-1},
\end{aligned} \tag{4.41}$$

where we used Lemma 3.3. Taking the limit $\lambda' \to 1$ in (4.41), we get that for pure TOA graphs $\overline{P}$ is finite. □

Theorem 4.16 can be generalized

Corollary 4.17. *Let $\mathcal{G}$ be a physical graph with a subset $V' \subset V$, $\|V'\| > 0$ such that for all $S' \subseteq V'$, $\|S'\| > 0$, $\lim_{\lambda\to 1} \overline{\chi(S')\widetilde{F}(\lambda)} \leq \|S'\|$, then*

$$\lim_{\lambda\to 1} \overline{\chi(S')\widetilde{P}(\lambda)} < \infty. \quad \forall S' \subseteq V',\ \|S'\| > 0. \tag{4.42}$$

4.4 Separability and statistical independence

Here we prove and discuss an important property characterizing mixed TOA graphs which introduces some simplifications in the study of statistical models on these very inhomogeneous structures. In this case, the graph $\mathcal{G}$ can be always decomposed in a pure TOA subgraph $\mathcal{S}$ and an ROA subgraph $\overline{\mathcal{S}}$ with independent jumping probabilities by cutting out a zero measure set of edges. The separability property implies that the two subgraphs are statistically independent and that their thermodynamic properties can be studied separately. Indeed, the partition functions of magnetic models referring to the two subgraphs factorize [11]. Let us first prove the following lemma.

Lemma 4.18. *The vertices of any mixed TOA graph can be divided into two subsets $V', \overline{V'} \subset V$ with $\|V'\|, \|\overline{V'}\| > 0$ and $\|\partial V'\| = \|\partial\overline{V'}\| = 0$, and such that*

$$\lim_{\lambda\to 1^-} \frac{\overline{\chi(S')\widetilde{F}(\lambda)}}{\|S'\|} < 1 \tag{4.43}$$

for all $S' \subseteq V'$ *with* $\|S'\| > 0$, *and*

$$\lim_{\lambda \to 1^-} \frac{\overline{\chi(S'')\widetilde{F}(\lambda)}}{\|S''\|} = 1 \tag{4.44}$$

for all $S'' \subseteq \overline{V'}$ *with* $\|S''\| > 0$.

Proof. From Definition 4.13 we have that a mixed TOA graph can always be decomposed into two subsets V' and $\overline{V'}$ satisfying (4.43) and (4.44). Now we show that $\|\partial V'\| = \|\partial \overline{V'}\| = 0$. Let us suppose that $\|\partial V'\| > 0$, from the boundedness condition on z_i we have $z_{\max}^{-1}\|\partial V'\| \leq \|\partial \overline{V'}\| \leq z_{\max}\|\partial V'\|$, and also $\|\partial \overline{V'}\| > 0$. Then from Corollaries 4.12 and 4.17 we get:

$$\lim_{\lambda \to 1} \overline{\chi(\partial V')\widetilde{P}(\lambda)} \leq \infty \tag{4.45}$$

and

$$\lim_{\lambda \to 1} \overline{\chi(\partial \overline{V'})\widetilde{P}(\lambda)} = \infty. \tag{4.46}$$

We will show that

$$\lim_{\lambda \to 1^-} \overline{\chi(\partial V')\widetilde{P}(\lambda)} \geq z_{\max}^{-2} \lim_{\lambda \to 1^-} \overline{\chi(\partial \overline{V'})\widetilde{P}(\lambda)}.$$

Then the hypotheses $\|\partial V'\| > 0$ would lead to a contradiction, proving that $\|\partial V'\| = \|\partial \overline{V'}\| = 0$. Let us evaluate $\widetilde{P}_i(\lambda)$ at a site $i \in \partial V'$:

$$\widetilde{P}_i(\lambda) = \sum_t \lambda^t p_{ii}^t = \sum_t \lambda^t \sum_{jk} p_{ik} p_{kj}^{t-2} p_{ji} \geq \sum_t \lambda^t \sum_{j \in \partial \overline{V'}} p_{ij} p_{jj}^{t-2} p_{ji}, \tag{4.47}$$

where in the inequality we do not consider the terms in which $j \neq k$ and $j \notin \partial \overline{V'}$. Exploiting the fact that $p_{ij} \geq 1/z_{\max}$ we get:

$$\widetilde{P}_i(\lambda) \geq \frac{\lambda^2}{z_{\max}^2} \sum_t \lambda^{t-2} \sum_{j \in S_{i,\partial \overline{V'}}} p_{jj}^{t-2} = \frac{\lambda^2}{z_{\max}^2} \sum_{j \in S_{i,\partial \overline{V'}}} \widetilde{P}_j(\lambda), \tag{4.48}$$

where $S_{i,\partial \overline{V'}} = \{j \in \partial \overline{V'} | \exists (i, j) \in E\}$. By averaging over the sites $i \in \partial V'$ we obtain:

$$\begin{aligned} \overline{\chi(\partial V')\widetilde{P}(\lambda)} &\geq \frac{\lambda^2}{z_{\max}^2} \lim_{r \to \infty} \frac{\chi_i(\partial V')}{|V_r|} \sum_{i \in V_r} \sum_{j \in S_{i,\partial \overline{V'}}} \widetilde{P}_j(\lambda) \\ &\geq \frac{\lambda^2}{z_{\max}^2} \lim_{r \to \infty} \frac{\chi_j(\partial \overline{V'})}{|V_r|} \sum_{j \in V_r} \widetilde{P}_j(\lambda) = \frac{\lambda^2}{z_{\max}^2} \overline{\chi(\partial \overline{V'})\widetilde{P}(\lambda)}. \end{aligned} \tag{4.49}$$

Taking the limit $\lambda \to 1$ we have:

$$\lim_{\lambda\to 1} \overline{\chi(\partial V')\widetilde{P}(\lambda)} \geq \frac{1}{z_{\max}^2} \lim_{\lambda\to 1} \overline{\chi(\partial \overline{V'})\widetilde{P}(\lambda)} \tag{4.50}$$

□

Theorem 4.19. *Let $\mathcal{G}$ be a mixed TOA graph. It is always possible to find two subgraphs $\mathcal{G}'$ and $\overline{\mathcal{G}'}$ of $\mathcal{G}$ with the following properties (we denote by $V' \subset V$, $\overline{V'} \subset V$, $E' \subset E$ and $\overline{E'} \subset E$ the positive measure sets of vertices and links defining $\mathcal{G}'$ and $\overline{\mathcal{G}'}$, respectively):*

1. *$\mathcal{G}'$ is pure TOA;*
2. *$\overline{\mathcal{G}'}$ is ROA;*
3. *$\|V'\| > 0$, $\|\overline{V'}\| > 0$;*
4. *$V' \cap \overline{V'} = \emptyset$, $V' \cup \overline{V'} = V$;*
5. *$\|E_{\mathcal{G}',\overline{\mathcal{G}'}}\| = 0$, where $E_{\mathcal{G}',\overline{\mathcal{G}'}} = \{(i,j) \in E \mid (i,j) \notin E', (i,j) \notin \overline{E'}\}$ is the set of links one has to cut for disconnecting the two subgraphs.*

Proof. Let us choose the vertex sets $V' \subset V$ and $\overline{V'} \subset V$ as the two subsets satisfying conditions (4.43) and (4.44) of Lemma 4.18, respectively. We then have $V' \cap \overline{V'} = \emptyset$, $V' \cup \overline{V'} = V$, $\|V'\| > 0$ and $\|\overline{V'}\| > 0$. Defining the sets of edges $E' = \{(i,j) \in E \mid i \in V', j \in V'\}$ and $\overline{E'} = \{(i,j) \in E \mid i \in \overline{V'}, j \in \overline{V'}\}$, one obtains $E_{\mathcal{G}',\overline{\mathcal{G}'}} = \{(i,j) \in E \mid i \in V', j \in \overline{V'}\}$, then from the boundedness of z_i, $\|E_{\mathcal{G}',\overline{\mathcal{G}'}}\| \leq z_{\max}\|\partial V'\| = 0$. Now we have to prove that $\mathcal{G}'$ is a pure TOA graph and $\overline{\mathcal{G}'}$ is an ROA graph with respect to their own transition probabilities $p_{ij}^{\mathcal{G}'}$ and $p_{ij}^{\overline{\mathcal{G}'}}$. We denote the average and the measure evaluated in $\mathcal{G}'$ by adding the superscript $\mathcal{G}'$. Then the following simple relation holds:

$$\overline{\phi'}^{\mathcal{G}'} = \|V'\|^{-1}\overline{\chi(V')\phi}, \tag{4.51}$$

where ϕ is any extension to V of the function $\phi' : V' \to \mathbb{R}$; in particular one has $\|\cdot\|^{\mathcal{G}'} = \|V'\|^{-1}\|\cdot\|$. Putting $\widetilde{V}'_{\partial V',t} = \{i \in V' | d(i,\partial V') \leq t\}$ (Definition 2.2) we get from the boundedness of the coordination number:

$$\|\widetilde{V}'_{\partial V',t}\|^{\mathcal{G}'} = \|V'\|^{-1}\|\widetilde{V}'_{\partial V',t}\| < \|V'\|^{-1}(z_{\max})^t\|\partial V'\| = 0; , \tag{4.52}$$

because $\|\partial V'\| = 0$ (Lemma 4.18). Let S' be any subset of V'. For the average of $F_{ii}^{\mathcal{G}'}(t)$ we have ($\overline{\widetilde{V}}'_{\partial V',t}$ is the complement of $\widetilde{V}'_{\partial V',t}$ in V'):

$$\begin{aligned} \overline{\chi(S')F^{\mathcal{G}'}(t)}^{\mathcal{G}'} &= \overline{\chi(\overline{\widetilde{V}}'_{\partial V',t})\chi(S')F^{\mathcal{G}'}(t)}^{\mathcal{G}'} + \overline{\chi(\widetilde{V}'_{\partial V',t})\chi(S')F^{\mathcal{G}'}(t)}^{\mathcal{G}'} \\ &= \overline{\chi(\overline{\widetilde{V}}'_{\partial V',t})\chi(S')F^{\mathcal{G}'}(t)}^{\mathcal{G}'} . \end{aligned} \tag{4.53}$$

Since $F_{ii}^{\mathcal{G}'}(t) = F_{ii}(t)$ on $\overline{\widetilde{V}}'_{\partial V',t}$, we get

$$\begin{aligned} \overline{\chi(S')F^{\mathcal{G}'}(t)}^{\mathcal{G}'} &= \overline{\chi(\overline{\widetilde{V}}'_{\partial V',t})\chi(S')F(t)}^{\mathcal{G}'} \\ &= \overline{\chi(S')F(t)}^{\mathcal{G}'} = \|V'\|^{-1}\overline{\chi(S')F(t)}, \end{aligned} \tag{4.54}$$

where in the second equality we used $\|\overline{\widetilde{V}}'_{\partial V',t}\|^{\mathcal{G}'} = 1$, while in the last one we used (4.51) and the fact that $\chi(S')\chi(V') = \chi(S')$, since $S' \subseteq V'$. From Lemma 4.4 we get $\forall S' \subseteq V'$ and $\forall \lambda < 1$:

$$\begin{aligned} \left(\|S'\|^{\mathcal{G}'}\right)^{-1}\overline{\chi(S')\widetilde{F}^{\mathcal{G}'}(\lambda)}^{\mathcal{G}'} &= \left(\|S'\|\|V'\|^{-1}\right)^{-1}\sum_{t=0}^{\infty}\lambda^t\overline{\chi(S')F^{\mathcal{G}'}(t)}^{\mathcal{G}'} \\ = \left(\|S'\|\|V'\|^{-1}\right)^{-1}\sum_{t=0}^{\infty}\lambda^t\|V'\|^{-1}\overline{\chi(S')F(t)} &= \|S'\|^{-1}\overline{\chi(S')\widetilde{F}(\lambda)}. \end{aligned} \tag{4.55}$$

Taking the limit $\lambda \to 1^-$ from (4.43) we obtain that $\mathcal{G}'$ is a pure TOA graph. In an analogous way one can prove that $\overline{\mathcal{G}'}$ is an ROA graph. □

Remark 4.20. Properties similar to the separability discussed in this section can also be found when considering the average spectral dimension instead of the simple recurrence or transience. This leads to the introduction of the so called *spectral classes* and *spectral subclasses*, which have been studied in details in the physical literature in connection with the problem of critical phenomena on graphs [8, 9]. We refer the reader to [8] for details, where the existence of the spectral dimension for classes and subclasses, as usual in physical papers, is implicitly assumed.

5 Harmonic oscillations

5.1 The physical model

Let us consider a countable system of particles $i \in \mathbb{N}$ interacting pairwise with a harmonic potential. In the simplest case in which all particles have the same mass m and their position can be described by a scalar $x_i(t) \in \mathbb{R}$ ($t \in \mathbb{R}$ is the time), we have that the motion equations for the system are:

$$m\frac{d^2x_i(t)}{dt^2} = \sum_j J_{ji}(x_j(t) - x_i(t)) = -\sum_j L_{ji}x_j(t), \tag{5.56}$$

where J_{ji} (Definition 1.5) represent the elastic constants describing the interaction between particles i and j. When all particles interact with the same strength k, we get:

$$m\frac{d^2x_i(t)}{dt^2} = \sum_j kA_{ji}(x_j(t) - x_i(t)) = -\sum_j k\Delta_{ji}x_j(t). \tag{5.57}$$

Equation (5.57) has been widely studied in physics to describe the elastic and thermal properties of solids. Solving (5.56) and (5.57) can be reduced to an eigenvalue problem by standard differential equation techniques. Denoting by $\widetilde{x}_i(\omega)$ the Fourier transform of $x_i(t)$ with respect to time t, (5.57) becomes:

$$\omega^2\widetilde{x}_i(\omega) = \frac{k}{m}\sum_j \Delta_{ji}\widetilde{x}_j(\omega) \tag{5.58}$$

Hence the study of the spectral properties of the Laplacian matrix Δ_{ij} plays a fundamental role in understanding the physical properties of harmonic oscillations.

5.2 The spectrum of the Laplacian

For the infinite graph X, the Laplacian Δ can be considered as a linear operator on the Hilbert space $l^2(V_X)$. From this point of view many rigorous results have been proven (see [26] for a review). An important theorem, giving a bound on the harmonic frequencies, is:

Theorem 5.1. *Let* $\mathrm{spec}(\Delta)$ *be the spectrum of the operator* $\Delta : l^2(V_X) \to l^2(V_X)$ *on a graph X with bounded connectivity (satisfying* **p.c.2***). Then* $\mathrm{spec}(\Delta) \subseteq [0, 2z_{\max}]$.

From a physical point of view, it is more useful to explore the graph using the Van Hove spheres $\mathcal{S}_{o,r}$. In particular, we are interested in the study of the low frequency (infrared) spectrum of the Laplacian, since many properties, as low temperature vibrational specific heat, are strictly related to the behaviour of this spectral region.

Definition 5.2. Given a physical graph $\mathcal{G}$, let $\Delta_{ij}^{o,r}$ be the Laplacian matrix relative to $\mathcal{S}_{o,r}$, and let $N_{o,r}(\epsilon)$ be the number of eigenstates of $\Delta_{ij}^{o,r}$ in the interval $[0, \epsilon]$. We define the integrated density of states as:

$$n(\epsilon) = \lim_{r\to\infty} n_{o,r}(\epsilon) = \lim_{r\to\infty} |V_{o,r}|^{-1} N_{o,r}(\epsilon). \tag{5.59}$$

Remark 5.3. The general conditions for the existence and the independence from o of limit (5.59) is another interesting mathematical open problem. In physics the existence and the independence are always assumed.

Definition 5.4. Given a physical graph $\mathcal{G}$, we say that $n(\epsilon)$ has a polynomial infrared behaviour if $\exists c_1, c_2, d_\omega \in \mathbb{R}^+$ such that

$$c_1\epsilon^{d_\omega/2} \le n(\epsilon) \le c_2\epsilon^{d_\omega/2}. \tag{5.60}$$

Remark 5.5. Heuristic results put into evidence that if d_ω is well-defined, then $\overline{d}$ is also well-defined and $d_\omega = \overline{d}$. This point also needs a rigorous mathematical formulation.

6 The Gaussian model

6.1 The model

A simple statistical model describing the average properties of harmonic oscillators is given by the Gaussian model. We first define it for a finite graph and then study the behaviour on an infinite structure using the Van Hove spheres.

Definition 6.1. Given a finite graph X in which each site represents a particle i, let $x_i \in \mathbb{R}$ be the displacement from the particle equilibrium position, and let $k_i \in \mathbb{R}$ ($k_i > 0$) and J_{ij} (Definition 1.5) represent the elastic constants describing the recoil force towards the equilibrium position and the interaction with the nearest neighbour particle j, respectively. The Hamiltonian of the system is

$$H = \frac{1}{4} \sum_{i,j \in X} J_{ij}(x_i - x_j)^2 + \frac{1}{2} \sum_{i \in X} k_i x_i^2 = \frac{1}{2} \sum_{i,j \in X} (L_{ij} + K_{ij}) x_i x_j, \qquad (6.61)$$

where $K_{ij} = k_i \delta_{ij}$, and $L_{ij} + K_{ij}$ is called the Hamiltonian matrix. Given a function $f = f(x_1, x_2, \ldots, x_{|V_X|}) : \mathbb{R}^{|V_X|} \to \mathbb{R}$ of the displacements x_i, we define the Boltzmann average of f as

$$\langle f \rangle_X(J, K) = \int f d\mu_X(x), \qquad (6.62)$$

where

$$d\mu(x) = Z^{-1} e^{-H} \prod_{i \in X} dx_i \quad \text{and} \quad Z = \int e^{-H} \prod_{i \in X} dx_i. \qquad (6.63)$$

We denote the Boltzmann averages simply by $\langle f \rangle_X$ (dropping (J, K)), when it is not necessary to evidence the dependence on some specific couplings. Since the Hamiltonian matrix is a positive defined operator, the Boltzmann average is well-defined for all continuous bounded functions. For infinite graphs, we denote by K_{ij} the local coupling matrix $K_{ij} = k_i \delta_{ij}$, $k_i \in \mathbb{R}^+$ ($k_{\max} > k_i > k_{\min}, \forall i \in V, k_{\max}, k_{\min} \in \mathbb{R}^+$).

Definition 6.2. Given a physical graph $\mathcal{G}$, a ferromagnetic coupling matrix J_{ij} and a local coupling matrix K_{ij}, let $\mathcal{S}_r$ be a sequence of Van Hove spheres, and let J^r_{ij} and K^r_{ij} be the matrices on $\mathbb{R}^{|V_r|}$ defined as $J^r_{ij} = J_{ij}$, $K^r_{ij} = K_{ij}$ if $i, j \in V_r$, and $J^r_{ij} = K^r_{ij} = 0$ otherwise. If $f_r : \mathbb{R}^{|V_r|} \to \mathbb{R}$ is a function of the displacements x_i

$i \in V_r$, we denote by $\langle f_r \rangle_r$ the Boltzmann average of f_r (6.62) defined in $\mathcal{S}_r$ by J_{ij}^r and by the constants k_i. The Boltzmann average of f on the graph $\mathcal{G}$ is then

$$\langle f \rangle = \lim_{r\to\infty} \langle f_r \rangle_r . \tag{6.64}$$

In physics the most interesting function f_r is the two point correlation function $x_i x_j$ representing the response of the system in the site j to an excitation in the site i, and the square average displacement:

$$\overline{x^2}^r = |V_r|^{-1} \sum_{i\in V_r} x_i^2 \tag{6.65}$$

Another interesting quantity is the many points correlation function $x_{i_1} x_{i_2} \dots x_{i_n}$. However, for the Gaussian model the average of these functions can be evaluated as:

$$\langle x_{i_1} x_{i_2} \dots x_{i_n} \rangle_r = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \frac{1}{n!!} \sum_{k_1 \dots k_n = 1}^{n} p^{k_1 \dots k_n} \langle x_{i_{k_1}} x_{i_{k_2}} \rangle_r \dots \langle x_{i_{k_{n-1}}} x_{i_{k_n}} \rangle_r , & \text{if } n \text{ is even,} \end{cases} \tag{6.66}$$

where $p^{k_1 \dots k_n}$ is the tensor of index permutations. Hence $\langle x_{i_1} x_{i_2} \dots x_{i_n} \rangle_r$ can be reduced to the two points correlation function.

Remark 6.3. For Definition 6.2 the main open problem also regards the conditions on the graph $\mathcal{G}$ and on the functions f_r guaranteeing the existence of the limit as $r \to \infty$. In the following we will prove the existence of the limit for the two points correlation functions, and then use (6.66) for $\langle x_{i_1} x_{i_2} \dots x_{i_n} \rangle_r$. On the other hand, a proof for the square average displacement is still lacking. For $\langle x_{i_1} x_{i_2} \dots x_{i_n} \rangle_r$ and $\langle \overline{x^2} \rangle^r$ we will prove also the independence from the center of the spheres o.

Let us introduce an alternative definition for the Gaussian model [24].

Definition 6.4. Given a graph X, a ferromagnetic coupling matrix J_{ij}, and a local coupling matrix K_{ij}, there exists a unique Gaussian probability measure $d\mu_g(x)$ on $l^\infty(V)$ with mean zero and covariance $(L + K)^{-1}$, see [24]. The measure $d\mu_g(x)$ characterizes the Gaussian model, and we will use the notations

$$\langle f(x) \rangle_g = \int f(x) d\mu_g(x), \tag{6.67}$$

and, in particular,

$$\langle x_i x_j \rangle_g = (L + K)_{ij}^{-1}. \tag{6.68}$$

In [24] some interesting properties of the linear operator $(L+K)^{-1}$ on $l^\infty(V)$ are obtained. These properties trivially hold also for the operator $(L^r + K^r)^{-1}$ on $\mathbb{R}^{|V_r|}$, where $L^r = I_i^r \delta_{ij} - J_{ij}^r$, $I_i^r = \sum_j J_{ij}^r$. In particular,

Lemma 6.5. *The operators $(L+K)^{-1}$ on $l^{\infty}(V)$ and $(L^r+K^r)^{-1}$ on $\mathbb{R}^{|V_r|}$ are positive bounded, and*

$$\|(L+K)^{-1}\| \le k_{\min}^{-1}, \quad \|(L^r+K^r)^{-1}\| \le k_{\min}^{-1}. \tag{6.69}$$

6.2 The walk expansion

Let us first consider the two points correlation functions.

Lemma 6.6. *Let J_{ij} be a ferromagnetic coupling matrix, and K_{ij} be a local coupling matrix on a graph X. The two points correlation function can be evaluated in the following way:*

$$\langle x_i x_j\rangle_g = \left(\left(1+\frac{z_{\max}J_{\max}}{k_{\min}}\right)k_j\right)^{-1}\widetilde{P}^J_{ij}\left(\left(1+\frac{k_{\min}}{z_{\max}J_{\max}}\right)^{-1}\right), \tag{6.70}$$

where $\widetilde{P}^J_{ij}(\lambda)$ is the generating function of the random walk defined by the jumping probability p^I_{ij}:

$$p^I_{ij} = \frac{k_{\min}}{z_{\max}J_{\max}k_i}J_{ij} + \left(1 - I_j\frac{k_{\min}}{z_{\max}J_{\max}k_i}\right)\delta_{ij}. \tag{6.71}$$

Proof. Let us represent $\langle x_i x_j\rangle_g$ by a walk expansion:

$$\begin{aligned}
\langle x_i x_j\rangle_g &= (I-J+K)^{-1}_{ij}\\
&= \left(\left(1+\frac{z_{\max}J_{\max}}{k_{\min}}\right)K\left(\delta - \frac{J+\frac{z_{\max}J_{\max}}{k_{\min}}K - I}{\left(1+\frac{k_{\min}}{z_{\max}J_{\max}}\right)\frac{z_{\max}J_{\max}}{k_{\min}}K}\right)\right)^{-1}_{ij}\\
&= \left(\left(1+\frac{z_{\max}J_{\max}}{k_{\min}}\right)k_j\right)^{-1}\left(\delta - \frac{J+\frac{z_{\max}J_{\max}}{k_{\min}}K - I}{\left(1+\frac{k_{\min}}{z_{\max}J_{\max}}\right)\frac{z_{\max}J_{\max}}{k_{\min}}K}\right)^{-1}_{ij}\\
&= \left(\left(1+\frac{z_{\max}J_{\max}}{k_{\min}}\right)k_j\right)^{-1}\sum_{t=0}^{\infty}\left(1+\frac{k_{\min}}{z_{\max}J_{\max}}\right)^{-t}P^I_{ij}(t),
\end{aligned} \tag{6.72}$$

where $P^I_{ij}(t) = (p^I)^t_{ij}$. Notice that p^I_{ij} is a well-defined transition probability with $0 \le p^I_{ij} \le 1$ and $\sum_j p^I_{ij} = 1$, then also $0 \le P^I_{ij}(t) \le 1$. From (6.72) one immediately gets (6.70). □

Corollary 6.7. *The two points correlation function $\langle x_i x_j \rangle_g$ satisfies the inequality*

$$0 \leq \langle x_i x_j \rangle_g \leq k_j^{-1}. \tag{6.73}$$

Proof. Equation (6.73) is a simple consequence of Lemma 6.6 and Lemma 3.3. □

Theorem 6.8. *Let $\mathcal{G}$ be a physical graph , J_{ij} be a ferromagnetic coupling matrix and K_{ij} be a local coupling matrix. For the two points correlation function we have:*

$$\lim_{r\to\infty} \langle x_i x_j \rangle_r = \langle x_i x_j \rangle_g. \tag{6.74}$$

Proof. For the correlation function $\langle x_i x_j \rangle_r$ in analogy to (6.72) we get the equation

$$\begin{aligned} \langle x_i x_j \rangle_r &= (L^r + K^r)^{-1}_{ij} \\ &= \left(\left(1 + \frac{z_{\max} J_{\max}}{k_{\min}}\right) k_j\right)^{-1} \sum_{t=0}^{\infty} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-t} P_{ij}^{I_r}(t), \end{aligned} \tag{6.75}$$

where $P_{ij}^{I_r}(t)$ is defined by the jumping probability $p_{ij}^{I_r}$ in $\mathcal{S}_{o,r}$:

$$p_{ij}^{I_r} = \frac{k_{\min}}{z_{\max} J_{\max} k_i} J_{ij}^r + \left(1 - I_j^r \frac{k_{\min}}{z_{\max} J_{\max} k_i}\right) \delta_{ij}. \tag{6.76}$$

Let us now choose spheres of radius $r(T) = r_{i,o} + T + 1$ ($T \in \mathbb{N}$), so that we can get the thermodynamic limit letting $T \to \infty$. We have:

$$\begin{aligned} \langle x_i x_j \rangle_{r(T)} = &\left(\left(1 + \frac{z_{\max} J_{\max}}{k_{\min}}\right) k_j\right)^{-1} \\ &\times \left(\sum_{t=0}^{T} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-t} P_{ij}^{I}(t) \right. \\ &\left. + \sum_{t=T+1}^{\infty} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-t} P_{ij}^{I_{r(T)}}(t) \right), \end{aligned} \tag{6.77}$$

where we used the property that $P_{ij}^{I_{r(T)}}(t) = P_{ij}^{I}(t)$ in $\mathcal{S}_{o,r(T)}$ for walks starting from i and of length smaller than $T+1$. Let us show that the second term in (6.77) goes to zero if $T \to \infty$, i.e., in the thermodynamic limit. Since $0 \leq P_{ij}^{I_{r(T)}}(t) \leq 1$, one gets:

$$\begin{aligned} 0 &\leq \left(\left(1 + \frac{z_{\max} J_{\max}}{k_{\min}}\right) k_j\right)^{-1} \sum_{t=T+1}^{\infty} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-t} P_{ij}^{I_{r(T)}}(t) \\ &\leq k_i^{-1} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-T} \longrightarrow 0 \ \text{ for } \ T \to \infty. \end{aligned} \tag{6.78}$$

Letting $T \to \infty$ in (6.77), from (6.78) we have

$$\begin{aligned}\langle x_i x_j \rangle &= \lim_{T\to\infty} \langle x_i x_j \rangle_{r(T)} \\ &= \left(\left(1 + \frac{z_{\max} J_{\max}}{k_{\min}}\right) k_j \right)^{-1} \sum_{t=0}^{\infty} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-t} P^I_{ij}(t) \qquad (6.79) \\ &= \langle x_i x_j \rangle_g. \end{aligned}$$

□

Corollary 6.9. $\langle x_i x_j \rangle = \lim_{r\to\infty} \langle x_i x_j \rangle_r$ *exists and is independent of the center of the spheres o, for all graphs X.*

Proof. Since $\langle x_i x_j \rangle_g$ is well-defined and independent of o, one immediately gets the claim from equation (6.79). □

When we deal with the thermodynamic limit we usually restrict ourselves to a physical graph. However, Definition 6.4 and Theorem 6.8 hold even for a graph which does not satisfy **p.c.1**, **p.c.2** and **p.c.3**. Hence the existence of $\langle x_i x_j \rangle$ and its the independence from o is proven for all graphs and not only for physical ones.

Corollary 6.10. *The many points correlation function*

$$\langle x_{i_1} x_{i_2} \dots x_{i_n} \rangle = \lim_{r\to\infty} \langle x_{i_1} \dots x_{i_n} \rangle$$

exists and is independent of the center of the spheres o, for all graphs X. Moreover, $\langle x_{i_1} x_{i_2} \dots x_{i_n} \rangle = \langle x_{i_1} x_{i_2} \dots x_{i_n} \rangle_g$.

Proof. Equations (6.66) hold for the averages $\langle\, . \,\rangle_r$ on any finite sphere and for the average $\langle\, . \,\rangle_g$ of Definition 6.4, hence from (6.79) we have $\langle x_{i_1} x_{i_2} \dots x_{i_n} \rangle = \langle x_{i_1} x_{i_2} \dots x_{i_n} \rangle_g$. Moreover, since $\langle x_{i_1} x_{i_2} \dots x_{i_n} \rangle_g$ are always well-defined and independent from o, one immediately completes the proof. □

Let us now pass to the study of the average displacement.

Theorem 6.11. *Let $\mathcal{G}$ be a physical graph, let J_{ij} be a ferromagnetic coupling matrix, and let K_{ij} be a local coupling matrix. For the average displacement we have*

$$\overline{\langle x^2 \rangle} = \lim_{r\to\infty} \langle \overline{x^2}^r \rangle_r = \overline{\langle x^2 \rangle_g}. \qquad (6.80)$$

Proof. In the hypothesis of the existence of $\overline{P^I(t)} = \lim_{r\to\infty} |V_r|^{-1} \sum_i P^I_{ii}(t)$ from (6.72) and Lemma 4.4 one has

$$\overline{\langle x^2 \rangle_g} = \left(\left(1 + \frac{z_{\max} J_{\max}}{k_{\min}}\right) k_j \right)^{-1} \sum_{t=0}^{\infty} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-t} \overline{P^I(t)}. \qquad (6.81)$$

Given a sequence of spheres $\mathcal{S}_r$, let $\widetilde{V}_{\partial V_r, t} = \{i \in V | d(i, \partial V_r) \le t\}$ (Definition 2.2) and $\overline{\widetilde{V}}_{\partial V_r, t}$ its complement, then from **p.c.2** we get

$$|\widetilde{V}_{\partial V_r, t}| \le z^T_{\max} |\partial V_r|, \qquad (6.82)$$

and from **p.c.3**

$$\lim_{r\to\infty} |V_r|^{-1} \sum_{i\in V_r} \chi_i(\widetilde{V}\partial V_r, t) = 0, \qquad \lim_{r\to\infty} |V_r|^{-1} \sum_{i\in V_r} \chi_i(\overline{\widetilde{V}}_{\partial V_r, t}) = 1. \tag{6.83}$$

The walk expansion (6.75) gives:

$$\begin{aligned}\lim_{r\to\infty} \langle \overline{x^2}^r \rangle_r = C_j \lim_{r\to\infty} \frac{1}{|V_r|} \sum_{i\in V_r} \Bigg(&\sum_{t=0}^{T} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-t} \chi_i(\overline{\widetilde{V}}_{\partial V_r, T}) P_{ii}^{I}(t) \\ &+ \sum_{t=0}^{T} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-t} \chi_i(\widetilde{V}_{\partial V_r, T}) P_{ii}^{I_r}(t) \\ &+ \sum_{t=T+1}^{\infty} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-t} P_{ii}^{I_r}(t) \Bigg),\end{aligned} \tag{6.84}$$

where we used the property that $P_{ii}^{I_r}(t) = P_{ii}^{I}(t)$ on $\overline{\widetilde{V}}_{\partial V_r, T}$ for $t \le T$ and the notation $C_j = (1 + z_{\max} J_{\max}/k_{\min})^{-1} k_j^{-1}$. Let us now evaluate the thermodynamic limit. From (6.83) and the boundedness of $P_{ii}^{I_r}(t)$ and $P_{ii}^{I}(t)$ we have

$$\begin{aligned}\lim_{r\to\infty} \langle \overline{x^2}^r \rangle_r = C_j \Bigg(&\sum_{t=0}^{T} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-t} \overline{P^I(t)} \\ &+ \lim_{r\to\infty} |V_r|^{-1} \sum_{i\in V_r} \sum_{t=T+1}^{\infty} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-t} P_{ii}^{I_r}(t) \Bigg).\end{aligned} \tag{6.85}$$

Since $0 \le P_{ij}^{I_{r(T)}}(t) \le 1$, one gets:

$$\begin{aligned} 0 &\le C_j \lim_{r\to\infty} |V_r|^{-1} \sum_{i\in V_r} \sum_{t=T+1}^{\infty} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-t} P_{ij}^{I_{r(T)}}(t) \\ &\le k_i^{-1} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-T} \longrightarrow 0 \ \text{ for } \ T \to \infty. \end{aligned} \tag{6.86}$$

Letting $T \to \infty$ in (6.85) from (6.86) we have:

$$\lim_{r\to\infty} \langle \overline{x^2}^r \rangle_r = C_j \sum_{t=0}^{\infty} \left(1 + \frac{k_{\min}}{z_{\max} J_{\max}}\right)^{-t} \overline{P^I(t)} = \overline{\langle x^2 \rangle_g}. \tag{6.87}$$

□

In the proof of Theorem 6.11 the hypotheses on the graph of satisfying **p.c.1**, **p.c.2**, **p.c.3** and on the existence of the thermodynamic limit are necessary, for example in the inhomogeneous Bethe lattice (Fig. 3) Definitions 6.2 and 6.4 are not equivalent.

Under these hypotheses equation (6.87) proves the independence of $\overline{\langle x^2 \rangle}$ from the choice of the center of the sphere, since $\overline{\langle x^2 \rangle}$ is reduced to the evaluation of averages of positive (bounded from below) functions.

6.3 Gaussian model and spectral dimensions

Lemma 6.6 puts into evidence the deep relation between the Gaussian model and random walks. An even clearer connection can be obtained if we consider the model defined by the Hamiltonian matrix $\Delta + mZ$ ($m \in \mathbb{R}^+$).

Theorem 6.12. *For the two points correlation function*

$$\langle x_i x_j \rangle (A, mZ) = \langle x_i x_j \rangle_g (\Delta, mZ) = z_j^{-1} \widetilde{P}_{ij} \left((1+m)^{-1} \right). \tag{6.88}$$

Proof. The first equation is a simple consequence of Theorem 6.8, while for the second one we have:

$$\begin{aligned} \langle x_i x_j \rangle_g (A, mZ) &= (Z - A + mZ)^{-1}_{ij} = z_j^{-1} (\delta - (1+m)^{-1} Z^{-1} A)^{-1}_{ij} \\ &= z_j^{-1} \sum_{t=0}^{\infty} (1+m)^{-t} P_{ij}(t). \end{aligned} \tag{6.89}$$

□

Equation (6.89) proves that $\langle x_i x_j \rangle_g (A, mZ)$ as a function of m is $\mathcal{C}^\infty$. Theorem 6.12 allows one to recast many graph properties which had been described using random walks in terms of correlation function of the Gaussian model.

Corollary 6.13. *A graph X is locally recurrent if and only if*

$$\lim_{m \to 0^+} \langle x_i x_i \rangle (A, mZ) = \infty.$$

Corollary 6.14. *A graph X is locally recurrent of degree N if and only if*

$$\lim_{m \to 0^+} \langle x_i x_i \rangle^{(n)} (A, mZ) < \infty, \ \forall n < N \quad \text{and} \quad \lim_{m \to 0^+} \langle x_i x_i \rangle^{(N)} (A, mZ) = \infty, \tag{6.90}$$

where $\langle x_i x_i \rangle^{(n)} (A, mZ) = (-1)^n (d^n / dm^n) \langle x_i x_i \rangle (A, mZ)$.

Corollary 6.15. *Let X be a graph recurrent of degree N with local spectral dimension* $\widetilde{d}$. *Then* $\widetilde{d} = 2(N - D + 1)$, *where*

$$D = \lim_{m \to 0^+} \frac{\log(\langle x_i x_i \rangle^{(N)} (A, mZ))}{-\log(m)}. \tag{6.91}$$

Proof. Corollaries 6.13, 6.14 and 6.15 are simple consequences of Theorem 6.12 and Definitions 3.4, 3.5 and 3.6. □

In [24] important universality properties are proven. Namely, it is shown that Corollaries 6.13, 6.14 and 6.15 hold even if one replaces Δ and Z with their generalizations L^r and K^r.

Let us now consider the behaviour of the average displacement.

Theorem 6.16. *Let $\mathcal{G}$ a physical graph. Then the average displacement satisfies the following equation*

$$\overline{\langle x^2\rangle}(A, mZ) = \overline{\langle x^2\rangle}(A, mZ) = \overline{z^{-1}\widetilde{P}\left((1+m)^{-1}\right)}. \tag{6.92}$$

Proof. Equation (6.92) is a consequence of Theorems 6.11 and 6.12. □

Corollary 6.17. *A physical graph $\mathcal{G}$ is recurrent on the average of degree N if and only if*

$$\lim_{m\to 0^+} \overline{\langle x^2\rangle}^{(n)}(A, mZ) < \infty,\ \forall n < N \quad \text{and} \quad \lim_{m\to 0^+} \overline{\langle x^2\rangle}^{(N)}(A, mZ) = \infty, \tag{6.93}$$

where $\overline{\langle x^2\rangle}^{(n)}(A, mZ) = (-1)^n (d^n/dm^n)\overline{\langle x^2\rangle}(A, mZ)$.

Corollary 6.18. *Let $\mathcal{G}$ be a physical graph recurrent on the average of degree N with average spectral dimension $\overline{d}$. Then $\overline{d} = 2(N - D + 1)$, where*

$$D = \lim_{m\to 0^+} \frac{\log(\overline{\langle x^2\rangle}^{(N)}(A, mZ))}{-\log(m)}. \tag{6.94}$$

Proof Corollaries 6.17 and 6.18 are simple consequences of Theorem 6.16, Definitions 4.6 and 4.7 and condition **p.c.2**, because in this case

$$z_{\max}^{-1}\overline{\widetilde{P}\left((1+m)^{-1}\right)} < \overline{z^{-1}\widetilde{P}\left((1+m)^{-1}\right)} < \overline{\widetilde{P}\left((1+m)^{-1}\right)}.$$ □

6.4 Universality properties of $\bar{d}$

In this section we will prove some important universality properties of the average spectral dimension, which have been stated in [6, 7]. Let us begin with

Lemma 6.19. *Let $\mathcal{G}$ be a physical graph. Then:*

$$\left(\frac{\partial^n}{\partial m^n}\right) \lim_{r\to\infty} \langle \overline{x^2}^r \rangle_r (J, mK) = \lim_{r\to\infty} \left(\frac{\partial^n}{\partial m^n}\right) \langle \overline{x^2}^r \rangle_r (J, mK). \tag{6.95}$$

Proof. From (6.75) one has:

$$\left(\frac{\partial^n}{\partial m^n}\right) \lim_{r\to\infty} \langle \overline{x^2}^r \rangle_r (J, mK)$$
$$= \left(\left(1 + \frac{z_{\max} J_{\max}}{k_{\min}}\right) k_j\right)^{-1} \left(\frac{\partial^n}{\partial m^n}\right) \lim_{r\to\infty} |V_r|^{-1}$$
$$\sum_{i\in V_r} \left(\sum_{t=0}^{T} \left(1 + \frac{m k_{\min}}{z_{\max} J_{\max}}\right)^{-t} P_{ii}^{I_r}(t) + \sum_{t=T+1}^{\infty} \left(1 + \frac{m k_{\min}}{z_{\max} J_{\max}}\right)^{-t} P_{ii}^{I_r}(t) \right), \tag{6.96}$$

where the probabilities $P_{ii}^{I_r}(t)$ obtained from (6.76) are independent of m. In the first term of (6.96), in the hypothesis of the existence of the thermodynamic limit we can exchange the limit and the derivatives, since this term is given by a sum of products of two functions, one independent of m, and the other one independent of r. Furthermore, using the property that $0 < P_{ii}^{I_r}(t) < 1$ one can prove that the second term tends to for $T \to \infty$. Hence, letting $T \to \infty$ we obtain (6.95). □

Theorem 6.20. *Let $\mathcal{G}$ be a physical graph, let J_{ij} be a ferromagnetic coupling matrix, and let K_{ij}, K'_{ij} be two local coupling matrices such that $k'_i \geq k_i$, $\forall i$. Then*

$$\langle \overline{x^2} \rangle^{(n)}(J, mK) \geq \langle \overline{x^2} \rangle^{(n)}(J, mK'). \tag{6.97}$$

Proof. Let us consider the sequence of spheres $\mathcal{S}_r$. Then

$$\frac{\partial}{\partial k_i} \langle \overline{x^2}^r \rangle_r^{(n)}(J^r, mK^r)$$
$$= \frac{-m}{|V_r|} \left[\frac{1}{L^r + mK^r} \left(K^r \frac{1}{L^r + mK^r}\right)^n \frac{1}{L^r + mK^r} \right]_{ii} \leq 0. \tag{6.98}$$

The square brackets in this formula contain a product of positive defined operators (Lemma 6.5). Hence,

$$\langle \overline{x^2}^r \rangle_r^{(n)}(J^r, mK^r) \geq \langle \overline{x^2}^r \rangle_r^{(n)}(J^r, mK'^r). \tag{6.99}$$

Using Lemma 6.19 and letting $r \to \infty$ we get (6.97). □

Corollary 6.21. *Let $\mathcal{G}$ be a physical graph recurrent of degree N and with average spectral dimension $\overline{d} = 2(N - D + 1)$. Then for any local coupling matrices K_{ij}*

$$\lim_{m\to 0^+} \langle \overline{x^2} \rangle^{(n)}(A, mK) < \infty,\ \forall n < N \quad \text{and} \quad \lim_{m\to 0^+} \langle \overline{x^2} \rangle^{(N)}(A, mK) = \infty, \tag{6.100}$$

and

$$D = \lim_{m\to 0^+} \frac{\log(\langle \overline{x^2} \rangle^{(N)}(A, mK))}{.} - \log(m) \tag{6.101}$$

Proof. From Theorem 6.20 and property **p.c.2** one has:

$$\langle \overline{x^2} \rangle^{(n)}(A, mk_{\max} Z) \leq \langle \overline{x^2} \rangle^{(n)}(A, mK) \leq \langle \overline{x^2} \rangle^{(n)}(A, mk_{\min} z_{\max}^{-1} Z). \tag{6.102}$$

From Corollaries 6.17, 6.18 and inequality (6.102) one gets (6.100) and (6.101). □

We proved the invariance of the average spectral dimension for any bounded rescaling of the local coupling matrix. Let us pass to examine rescalings of the ferromagnetic coupling matrix.

Theorem 6.22. *Let $\mathcal{G}$ be a physical graph , let K_{ij} be a local coupling matrix, and let J_{ij}, J'_{ij} be two ferromagnetic coupling matrices such that $J'_{ij} \geq J_{ij}, \forall (i, j) \in E$. Then*

$$\langle \overline{x^2} \rangle^{(n)}(J, mK) \geq \langle \overline{x^2} \rangle^{(n)}(J', mK). \tag{6.103}$$

Proof. Let us consider the sequence of spheres $\mathcal{S}_r$. Then

$$\begin{aligned} \frac{\partial}{\partial J_{ij}} \langle \overline{x^2}^r \rangle_r^{(n)}(J^r, mK^r) &= -\frac{1}{|V_r|} \Phi_{ii} + \Phi_{ji} - \Phi_{ij} - \Phi_{ji} \\ &= -\frac{1}{|V_r|} v^{(i,j)} \Phi v^{(i,j)} \leq 0, \end{aligned} \tag{6.104}$$

where $v \in \mathbb{R}^{|V_r|}$, $v_h^{(i,j)} = \delta_{ih} - \delta_{jh}$, and

$$\Phi_{hk} = \left[\frac{1}{L^r + mK^r} \left(K^r \frac{1}{L^r + mK^r} \right)^n \frac{1}{L^r + mK^r} \right]_{hk}. \tag{6.105}$$

The inequality in (6.104) holds since Φ (6.105) is a positive defined operator (Lemma 6.5). Therefore,

$$\langle \overline{x^2}^r \rangle_r^{(n)}(J^r, mK^r) \geq \langle \overline{x^2}^r \rangle_r^{(n)}(J'^r, mK^r). \tag{6.106}$$

By using Lemma 6.19 and letting $r \to \infty$, we get (6.103). □

Corollary 6.23. *Let $\mathcal{G}$ be a physical graph recurrent of degree N and with average spectral dimension $\bar{d} = 2(N - D + 1)$. Then for any ferromagnetic coupling matrix J_{ij} and any local coupling matrices K_{ij} we have:*

$$\lim_{m \to 0^+} \langle \overline{x^2} \rangle^{(n)}(J, mK) < \infty, \ \forall n < N \quad \text{and} \quad \lim_{m \to 0^+} \langle \overline{x^2} \rangle^{(N)}(J, mK) = \infty, \tag{6.107}$$

and

$$D = \lim_{m \to 0^+} \frac{\log(\langle \overline{x^2} \rangle^{(N)}(J, mK))}{-\log(m)}. \tag{6.108}$$

Proof. From Theorem 6.22 one has:

$$\begin{aligned} J_{\max}^{-1} \langle \overline{x^2} \rangle^{(n)}(A, mJ_{\max}^{-1} K) &\leq \langle \overline{x^2} \rangle^{(n)}(J, mK) \\ &\leq J_{\min}^{-1} \langle \overline{x^2} \rangle^{(n)}(A, mJ_{\min}^{-1} Z). \end{aligned} \tag{6.109}$$

From Corollaries 6.17, 6.18, 6.21 and inequality (6.109) one gets (6.107) and (6.108). □

So we have proven the invariance of the average spectral dimension for a bounded rescaling of the ferromagnetic couplings. In particular, from a random walks point of view we proved the invariance of $\overline{d}$ for a rescaling of the jumping probabilities given by (3.23). Let us pass to prove the invariance with respect to removing (or adding) a zero measure set of edges.

Theorem 6.24. *Given a physical graph $\mathcal{G}$, a local coupling matrix K_{ij}, a ferromagnetic coupling matrix J_{ij} and $E' \subset E$ such that $\|E'\|$, let $J'_{ij} = 0$ if $(i, j) \in E'$ and $J'_{ij} = J'_{ij}$ otherwise. Then*

$$\overline{\langle x^2 \rangle}(J, mK) = \overline{\langle x^2 \rangle}(J', mK). \tag{6.110}$$

Proof. Let us define the coupling matrix $J_{ij}(\alpha) = J_{ij}(1 - \alpha\chi_{(i,j)}(E'))$ ($\alpha \in \mathbb{R}$) so that $J_{ij}(0) = J_{ij}$ and $J_{ij}(1) = J'_{ij}$. Consider the sequence of increasing spheres $\mathcal{S}_r$ from (6.104):

$$\frac{\partial}{\partial\alpha}\langle \overline{x^2}^r\rangle_r(J^r(\alpha), mK^r) = \sum_{(i,j)\in E'_r} \frac{1}{|V_r|} J_{ij}(\alpha) v^{(i,j)} \Phi v^{(i,j)} \geq 0, \tag{6.111}$$

where $E'_r = \{(i, j) \in E' | \, i \in V_r, j \in V_r\}$ (Definition 2.9). From the boundedness of Φ ($\|\Phi\| \leq k_{\min}^{-1}$) one has

$$0 \leq \frac{\partial}{\partial\alpha}\langle \overline{x^2}^r\rangle_r(J^r(\alpha), mK^r) \leq 2J_{\max}k_{\min}^{-1}\frac{|E'_r|}{|V_r|}. \tag{6.112}$$

Integrating (6.112) on $\alpha \in [0, 1]$ we have:

$$0 \leq \langle \overline{x^2}^r\rangle_r(J^r(1), mK^r) - \langle \overline{x^2}^r\rangle_r(J^r(0), mK^r) \leq 2J_{\max}k_{\min}^{-1}\frac{|E'_r|}{|V_r|}. \tag{6.113}$$

Letting $r \to \infty$ in (6.113), and using the fact that $\|E'\| = 0$, we get

$$\overline{\langle x^2 \rangle}(J(1), mK) = \overline{\langle x^2 \rangle}(J(0), mK).$$

□

Theorem 6.25. *Given a physical graph $\mathcal{G}$ recurrent of degree N and with spectral dimension $\overline{d} = 2(N - D + 1)$, let $\mathcal{G}'$ be the graph given by $V' = V$ and $E' = \{(i, j)|(i, j) \in E \vee (i, j)|\exists k, (i, k), (k, j) \in E\}$ for all ferromagnetic coupling matrix J'_{ij} on $\mathcal{G}'$ and all local coupling matrices K'_{ij}. Then*

$$\lim_{m\to 0^+} \overline{\langle x^2 \rangle}^{(n)}(J', mK') < \infty, \ \forall n < N \quad \text{and} \quad \lim_{m\to 0^+} \overline{\langle x^2 \rangle}^{(N)}(J', mK') = \infty, \tag{6.114}$$

and

$$D = \lim_{m \to 0^+} \frac{\log(\langle \overline{x^2} \rangle^{(N)}(J', mK'))}{-\log(m)}. \tag{6.115}$$

Proof. From Corollaries 6.21 and 6.23 we have that it is enough to prove equations (6.114) and (6.115) for two particular matrices J'_{ij} and mK'_{ij}. In particular we can chose $mK'_{ij} = m\delta_{ij}$ and $J'_{ij} = A_{ij}(1 - \alpha(I_i + I_j)) + \alpha \sum_k A_{ik}A_{kj}$, where A_{ij} is a ferromagnetic coupling matrix on $\mathcal{G}$ and $\alpha \leq (2z_{\max})^{-1}$. With this condition it is easy to show that J'_{ij} is a well-defined ferromagnetic coupling matrix on $\mathcal{G}'$. Let $\mathcal{S}'_r$ be a sequence of increasing spheres in $\mathcal{G}'$ ($\mathcal{S}'_r$ correspond to spheres of radius $2r$ in $\mathcal{G}$ that will be called $\mathcal{S}_2r$), we have:

$$\begin{aligned} \langle \overline{x^2}^r \rangle_r^{(n)}(J'^r, K'^r) &= \frac{1}{|V'_r|} \sum_{i \in V'_r} \left(\frac{1}{L'^r + m\delta} \right)_{ii}^{n+1} \\ &= \frac{1}{|V_2r|} \sum_{i \in V_2r} \left(\frac{1}{\Delta^{2r} - \alpha(\Delta^{2r})^2 + m\delta} \right)_{ii}^{n+1}, \end{aligned} \tag{6.116}$$

where we used the property that the Laplacian corresponding to A'_{ij} is $\Delta^{2r} - \alpha(\Delta^{2r})^2$. If we evaluate expression (6.116) in the base where L is diagonal, denoting by $0 \leq l_k^{2r} \leq 2z_{\max}$ (Theorem 5.1) the eigenvalues of Δ^{2r} we get

$$\begin{aligned} \frac{1}{|V_2r|} \sum_k^{|V_2r|} \left(\frac{1}{l_k^{2r} + m} \right)^{n+1} &\leq \frac{1}{|V_2r|} \sum_k^{|V_2r|} \left(\frac{1}{l_k^{2r} - \alpha(l_k^{2r})^2 + m} \right)^{n+1} \\ &\leq \frac{1}{|V_2r|} \sum_k^{|V_2r|} \left(\frac{1}{l_k^{2r}(1 - \alpha 2z_{\max}) + m} \right)^{n+1} \end{aligned} \tag{6.117}$$

Hence, rewriting equation (6.117) in the base of the sites and letting $r \to \infty$, we get from Lemma 6.19:

$$\begin{aligned} \langle \overline{x^2} \rangle^{(n)}(A, m\delta) &\leq \langle \overline{x^2} \rangle^{(n)}(J', K') \\ &\leq (1 - \alpha 2z_{\max})^{-1} \langle \overline{x^2} \rangle^{(n)}(A, m(1 - \alpha 2z_{\max})^{-1}\delta). \end{aligned} \tag{6.118}$$

Inequality (6.118) proves equations (6.114) and (6.115). □

Applying Theorem 6.25 n times we have that $\overline{d}$ is invariant under addition (or removing) of couplings up to any finite distance n. Let us now introduce a very general transformation on the graph $\mathcal{G}$.

Definition 6.26. Let $\mathcal{G}$ be a physical graph, a topological rescaling of $\mathcal{G}$ is any graph $\mathcal{G}^P$ (defined by V^P and E^P) obtained by the following steps.

- Let $P = \{\mathcal{G}_m, \mathcal{G}_n, \dots\}$ any infinite partition of $\mathcal{G}$ given by the subgraphs $\mathcal{G}_n$ (defined by V_n and E_n) satisfying the properties $\forall n$ $\mathcal{G}_n$ is connected, $\bigcup_n^\infty V_n =$

V, $\forall n, m$ $V_n \cap V_m = \emptyset$, $\exists K \in \mathbb{N}$ such that $|V_n| < K$ $\forall n$ and $E_n = \{(i, j) \in E | i, j \in V_n\}$.

- $V^P = \{n | \exists \mathcal{G}_n \in P\}$.
- $E^P = \{(n, m) | \exists (i, j) \in E, i \in V_n, j \in V_m\}$

Theorem 6.27 ([7]). *Given a physical graph* $\mathcal{G}$ *its average spectral dimension* $\overline{d}$ *is invariant with respect to topological rescalings introduced in Definition 6.26.*

Proof. Any topological rescaling can be considered as a finite range transformation, the distance involved in the transformations is bounded by the maximum size of the subgraphs $\mathcal{G}_n$. Hence from Theorem 6.25 one immediately obtains the proof. □

7 Conclusions

The results we presented in the previous sections are the background common to any mathematical-physicist working on physical models and random walks on infinite graphs.

All these topics have direct physical applications in the physics of matter, which are explained in detail in the quoted literature.

Indeed, some more advanced topics, mainly concerning phase transitions and critical phenomena, have not been discussed here. This choice is due to two main reasons: they require specific technical knowledge of the general problem of phase transitions in physics and most of these results are to be considered simply as heuristic investigations.

We refer the interested reader to [33] for a general mathematical introduction to phase transitions and critical phenomena and to [9, 11, 12, 14, 15, 16, 25] for specific results on infinite graphs.

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