Anomalous diffusion and Hall effect on comb lattices

Raffaella Burioni,[1] Davide Cassi,[1] Giovanni Giusiano,[2] and Sofia Regina[1]
[1] *Istituto Nazionale di Fisica della Materia—INFM, Dipartimento di Fisica, Università di Parma, Parco Area delle Scienze 7A, 43100 Parma, Italy*
[2] *Istituto Nazionale di Fisica della Materia—INFM, Dipartimento di Fisica, Università di Perugia, Via A. Pascoli, 06123 Perugia, Italy*
(Received 8 October 2002; published 28 January 2003)

In this paper we study the effects of a magnetic field on the discrete time random walk of a classical charged particle moving on a comb lattice. We develop an analytical technique to study the Lorentz force effects on the asymptotic diffusion laws. This approach also allows the description of the combined action of an electric and a magnetic field (Hall effect). The generalization to other comblike branched structures is discussed.

DOI: 10.1103/PhysRevE.67.016116 PACS number(s): 05.50.+q, 05.40.Fb, 05.60.−k, 75.40.Gb

I. INTRODUCTION

Over the last decade a great deal of interest has been focused on the investigation of diffusion in disordered networks [1,2]. Diffusion on these structures does not exhibit the behavior typical of ordered crystalline lattices and it is often described by dramatically different laws. This is the case of a great variety of real systems such as percolation clusters, polymers, glasses, and fractals. Moreover, if the diffusing particles are electrically charged, the presence of external electric and magnetic fields can give rise to further anomalous phenomena [3–6].

The theoretical approach to the problem of diffusion in disordered structures is based on the study of random walks on nontranslationally invariant networks. The effects of an electric field are reproduced in the so-called biased random walk problem, where the probability of jumping in the direction of the field is greater than the probability of jumping in the opposite direction [7]. When a magnetic field is present, the Lorentz force must be taken into account. The simplest way to do this is to assume the random walker velocity to be a vector, having the direction of the last step of the walker and unitary length. According to this definition, the effect of the Lorentz force consists in changing the jumping probabilities in a point depending on the way followed by the walker to reach that particular point, i.e., the problem of random walks in presence of a magnetic field can be mapped into that of random walks with *short time memory*.

In this paper we develop an analytical technique to study diffusion of charged particles in disordered systems using biased random walks with short time memory (one-step memory). We apply our results to the particular case of the two-dimensional comb lattice in the presence of an electric field $\mathcal{E}$ and a magnetic field $\mathcal{B}$ (Fig. 1). This structure, in absence of external fields, is characterized by anomalous diffusion along the backbone since the average square displacement grows according to the relation $\langle x^2\rangle \sim t^{1/2}$ [1]. When an electric field is applied in the direction of the backbone, the walker is pushed by the field and the average displacement becomes $\langle x\rangle \sim t^{1/2}$. The combined application of an electric and a magnetic field, pushing the walker in opposite directions, gives rise to different situations depending on the relative strength of the two fields. We discuss the case of a magnetic field orthogonal to the comb, showing that the average displacement still grows as $\langle x\rangle \sim t^{1/2}$ with the particle following the direction of the electric field if $\mathcal{E} > \mathcal{B}/4$ and the opposite direction otherwise.

The paper is organized as follows. In Sec. II we present a review of the results concerning the diffusion on the comb lattices in the presence of an electric field [1], in Secs. III and IV we develop our technique of random walks with memory to study the effects of a magnetic field. Finally, in Sec. V we describe the combined effect of an electric and a magnetic field on a two-dimensional comb lattice.

II. DIFFUSION IN THE PRESENCE OF AN ELECTRIC FIELD

Let us briefly recall the results concerning diffusion on a comb lattice in the presence of an electric field applied in the direction of the backbone [1]. The comb lattice in Fig. 1 is a discrete structure consisting of a linear chain (backbone) whose points are connected with half-linear chains (teeth). We shall label each point of the backbone using a coordinate x, $x \in \mathbf{Z}$, and the points of the tooth linked with site x, with y_x, $y_x \in \mathbf{N}, y_x > 0$. Each point of the backbone is also connected to itself by a loop, representing a staying probability.

In absence of electric and magnetic fields, a random walker moving on the comb can jump from a generic point i to one of its z_i nearest neighbors with equal probability $1/z_i$. The staying probabilities, represented in Fig. 1 by the loops, change the coordination number z_i from $z_i = 3$ to $z_i = 4$. If

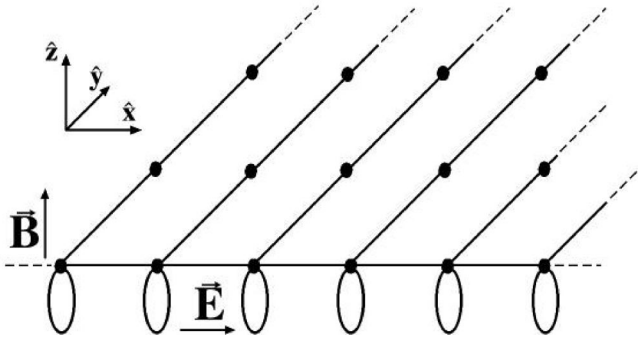

FIG. 1. The two-dimensional comb lattice in the presence of an electric field $\mathcal{E}$ and a magnetic field $\mathcal{B}$. Loops represent waiting probabilities.

we apply an electric field of strength $\mathcal{E}$, $|\mathcal{E}|<1/4$, the probability of jumping to the right becomes $1/4+\mathcal{E}$, the probability of jumping to the left is $1/4-\mathcal{E}$, while both the jumping probability on the teeth and the staying probability remain unchanged. Notice that we put $|\mathcal{E}|<1/4$ to prevent the jumping probabilities from assuming negative values. Let us start with the case $\mathcal{E}>0$. In this first case the walker is driven to the right (R) by the electric field. Diffusion along the backbone can be studied by evaluating the probability $P(O,x;t)$ of being on site x, $x>0$ after t steps, for a walker starting at $t=0$ from a fixed origin O on the backbone. The average displacement on the right of the origin O after t steps is given by

$$\langle x\rangle_R=\frac{\sum_{x=0}^{\infty} xP(O,x;t)}{\sum_{x=0}^{\infty} P(O,x;t)}. \tag{1}$$

We recall that, as usual, the average displacement after a time t is defined by

$$\langle x\rangle=\frac{\sum_{x=-\infty}^{\infty} xP(O,x;t)}{\sum_{x=-\infty}^{\infty} P(O,x;t)}.$$

In turn, $P(O,x;t)$ can be written as

$$P(O,x;t)=\sum_{m=0}^{t} F(O,x;m)P(x,x;t-m), \tag{2}$$

where $F(O,x;m)$ is the probability of being for the first time in point x after m steps for a walker starting at time $m=0$ from point O, and $P(x,x;t-m)$ is the probability of returning to x after $t-m$ steps. Thanks to the translational invariance of the lattice along the direction of the backbone we can introduce the notation $P(O;t)\equiv P(x,x;t)$, which holds for each point x of the backbone. The expression of $F(O,x;m)$ is given by

$$\begin{aligned}F(O,x;m)=&\left(\frac{1}{4}+\mathcal{E}\right)^x \sum_{m_0=0}^{\infty}\cdots\sum_{m_{x-1}=0}^{\infty}\\ &\times H_L(O;m_0)\cdots H_L(O;m_{x-1})\\ &\times\delta_{m,m_0+\cdots m_{x-1}-x},\end{aligned} \tag{3}$$

where $H_L(O;m_x)\equiv H_L(x,x;m_x)$ represents the probability of returning to the starting point on the backbone after m_x steps for a walker moving on a left half comb, i.e., on the structure that is obtained from the comb of Fig. 1 after suppressing all the points (and relative links) that occupy positions on the right of point x.

The previous relations can be written in terms of generating functions, using Tauberian theorems [8] to extract the asymptotic behavior for long times t of the probability functions from the corresponding generating functions. Using generating functions, Eqs. (2) and (3) become

$$\tilde{P}(O,x;\lambda)=\tilde{F}(O,x;\lambda)\tilde{P}(O;\lambda) \tag{4}$$

and

$$\tilde{F}(O,x;\lambda)=\left[\left(\frac{1}{4}+\mathcal{E}\right)\lambda\right]^x \tilde{H}_L(O;\lambda)^x. \tag{5}$$

The whole problem is now reduced to the determination of $\tilde{H}_L(O;\lambda)$ and $\tilde{P}(O,x;\lambda)$. Using the relations obtained in Ref. [9] for the case $\mathcal{E}=0$, we finally obtain for $\mathcal{E}\neq 0$

$$\tilde{P}(O;\lambda)=\frac{4}{\left(\sqrt{10-6\lambda-4\lambda^2+4\lambda^2\mathcal{E}^2+6\sqrt{1-\lambda^2}-2\lambda\sqrt{1-\lambda^2}}\right)^{1/2}}. \tag{6}$$

The asymptotic behavior of a probability function is determined by the singularities of the corresponding generating function calculated in the variable $\lambda=1-\epsilon$ in the limit $\epsilon\to 0$. In particular, if the k-order derivative of a generating function $\tilde{P}(i,j;1-\epsilon)$ diverges as $\epsilon^{-\sigma}(\sigma>0)$, the asymptotic behavior of $P(i,j;t)$ as $t\to\infty$ will be given by

$$P(i,j;t)\sim t^{\sigma-1-k},\quad t\to\infty, \tag{7}$$

i and j being two generic points of the lattice. Let us study in detail the asymptotic behavior of $\tilde{P}(O;\lambda)$ for $\mathcal{E}\neq 0$. It is known [10] that random walks are recursive (i.e., the walker returns to its initial position on the lattice with probability equal to 1) when

$$P(O;t)\sim t^{-\tilde{d}/2},\quad t\to\infty, \tag{8}$$

and the spectral dimension $\tilde{d}$ [11] is less or equal than 2. If $\tilde{d}>2$, the random walk is transient and the probability for the walker to reach the starting point is less than 1. Now, if $\mathcal{E}=0$, we know that $\tilde{P}(O;1-\epsilon)$ diverges as $\epsilon^{-1/4}$ [9]. This implies that

$$P(O;t)_{\mathcal{E}=0}\sim t^{-3/4},\quad t\to\infty, \tag{9}$$

and the spectral dimension of the comb is $\tilde{d}=3/2<2$. This result implies that unbiased random walks on the comb are recursive. On the contrary, if $\mathcal{E}\neq 0$,$\tilde{P}(O,1-\epsilon)$ does not diverge as $\epsilon\to 0$. To find a diverging quantity, we have to calculate the first-order derivative and we obtain

$$P(O;t)_{\mathcal{E}=0}\sim t^{-3/2},\quad t\to\infty, \tag{10}$$

so we have $\tilde{d}=3>2$ which represents a dramatic change from a regime of recursive random walks to a regime of transient. The electric field changes the spectral dimension causing a real transition in the system. Let us now consider $\langle x\rangle_R$ and its generating function:

$$\langle \tilde{x}(\lambda)\rangle_R = \frac{\sum_{x=0}^{\infty} x\tilde{P}(x;\lambda)}{\sum_{x=0}^{\infty} \tilde{P}(x;\lambda)} = \frac{\sum_{x=0}^{\infty} x\tilde{P}(O;\lambda)\tilde{H}_L(O;\lambda)^x\left[\left(\frac{1}{4}+\mathcal{E}\right)\lambda\right]^x}{\sum_{x=0}^{\infty} \tilde{P}(O;\lambda)\tilde{H}_L(O;\lambda)^x\left[\left(\frac{1}{4}+\mathcal{E}\right)\lambda\right]^x}. \quad (11)$$

These geometrical series can be easily summed, giving

$$\langle \tilde{x}(\lambda)\rangle = \frac{\tilde{P}(O;\lambda)\dfrac{\left(\frac{1}{4}+\mathcal{E}\right)\lambda\tilde{H}_L(O;\lambda)}{\left[1-\left(\frac{1}{4}+\mathcal{E}\right)\lambda\tilde{H}_L(O;\lambda)\right]^2}}{\tilde{P}(O;\lambda)\dfrac{1}{1-\left(\frac{1}{4}+\mathcal{E}\right)\lambda\tilde{H}_L(O;\lambda)}}; \quad (12)$$

in the limit $\lambda \to 1-\epsilon$ we have, for $\mathcal{E}\neq 0$,

$$\tilde{P}(O;\lambda) \to \frac{2}{\mathcal{E}} + O(\sqrt{\epsilon}) \quad (13)$$

and

$$\left(\frac{1}{4}+\mathcal{E}\right)\lambda\tilde{H}_L(O;\lambda) \to 1 + O(\sqrt{\epsilon}). \quad (14)$$

This implies that the denominator of Eq. (12) diverges as $1/\sqrt{\epsilon}$ while the numerator diverges as $1/\epsilon$. The asymptotic behavior is then given by

$$\langle x\rangle_R \sim 4\sqrt{2\pi}\mathcal{E}t^{1/2}, \quad t\to\infty. \quad (15)$$

The average displacement to the left of point O, $\langle x\rangle_L$, is obtained by substituting $\mathcal{E}$ with $-\mathcal{E}$ in Eq. (11). By straightforward calculation we find out that in this case,

$$\langle x\rangle_L \sim -\frac{2(1+4\mathcal{E})}{\mathcal{E}(3+4\mathcal{E})}, \quad t\to\infty. \quad (16)$$

For the diffusion along the direction of the teeth, since all the teeth are equivalent, the problem can be mapped into that of a particle diffusing on a semilinear chain having a staying probability equal to 3/4 in the origin. This comes from the fact that each time the walker reaches one of the joining points between tooth and backbone has a probability 1/4 of remaining on this same site, a probability $1/4-\mathcal{E}$ of reaching the joining point on the left and a probability $1/4+\mathcal{E}$ of reaching the joining point on the right. Since these threes point are equivalent from the point of view of the diffusion along the teeth, we can consider the sum of these three probabilities as a staying probability equal to 3/4. The average displacement on the half-linear chain is known to grow as $t^{1/2}$ [12], and it is not affected by the presence of a staying probability in the origin.

III. RANDOM WALKS IN A MAGNETIC FIELD

Let us now turn on a magnetic field of strength $\mathcal{B}$ directed along the z axis (the comb is in the x-y plane) and let us consider the effects of the Lorenz force on the walker. We define the random walker velocity in a generic point i as a vector having the direction of the last step of the walker, i.e., the step the walker took to reach that point. When the walker moves along one of the teeth, field $\mathcal{B}$ has no consequences since its only effect consists in trying to force the walker out of the structure, but this is forbidden. Only when the walker takes the final step from a tooth to the backbone, the field $\mathcal{B}$ modifies the jumping probabilities: pushing the charged particle to the right or to the left on the backbone depending on the sign of $\mathcal{B}$. The Lorentz force always comes into play when the walker moves along the backbone pushing the particle in the direction of the the teeth or towards the loops. Finally, the Lorentz force is null when the walker reaches the backbone coming from a loop, since in this case the velocity is supposed to be zero. A point x on the backbone can be reached in four different ways: from the left ($\rightarrow$), from the right ($\leftarrow$), from the tooth connected with point x ($\downarrow$), and after a waiting time on site x itself ($\uparrow$). Once reached point x, the walker experiences the Lorentz force while taking the next step. The Lorentz force will change the jumping probabilities depending on the way the walker reached point x. Let $p_1(\alpha)$, $p_2(\alpha)$, $p_3(\alpha)$, $p_4(\alpha)$, $\alpha=\rightarrow,\leftarrow,\uparrow,\downarrow$, be the probabilities of jumping right, jumping left, jumping on the tooth, and stay relative to a point on the backbone, respectively. We obtain the following jumping probabilities:

α	p_1	p_2	p_3	p_4
$\uparrow$	$\frac{1}{4}+\mathcal{E}$	$\frac{1}{4}-\mathcal{E}$	$\frac{1}{4}$	$\frac{1}{4}$
$\downarrow$	$\frac{1}{4}+\mathcal{E}-\mathcal{B}$	$\frac{1}{4}-\mathcal{E}+\mathcal{B}$	$\frac{1}{4}$	$\frac{1}{4}$
$\rightarrow$	$\frac{1}{4}+\mathcal{E}$	$\frac{1}{4}-\mathcal{E}$	$\frac{1}{4}-\mathcal{B}$	$\frac{1}{4}+\mathcal{B}$
$\leftarrow$	$\frac{1}{4}+\mathcal{E}$	$\frac{1}{4}-\mathcal{E}$	$\frac{1}{4}+\mathcal{B}$	$\frac{1}{4}-\mathcal{B}$

(17)

The jumping rules of the random walk are deeply modified by the magnetic field since the choice of the direction to follow is now influenced by the memory of the last step. In the following, every probability function will be written as

$$P^{\alpha}_{\beta}(t), \quad \alpha,\beta=\rightarrow,\leftarrow,\uparrow,\downarrow, \quad (18)$$

where β tells in which way the walker took the last step (corresponding to time t) and α represents the memory of the walker as it starts to move at $t=0$.

IV. DIFFUSION IN A MAGNETIC FIELD

In presence of a magnetic field, the diffusion of a random walker along the backbone of the comb can be still evaluated

using Eq. (1), but now memory effects must be taken into account. The probability that a random walker starting form the origin O at $t=0$ with an α-type memory arrives at a distance $x>0$ on the backbone after t steps is given by

$$P^{\alpha}(O,x;t)=\sum_{\beta} P^{\alpha}_{\beta}(O,x;t). \tag{19}$$

Moreover, x being on the right of the starting point, we can write

$$P^{\alpha}(O,x;t)=\sum_{m} F^{\alpha}_{\rightarrow}(O,x;m)P^{\rightarrow}(O;t-m) \tag{20}$$

and

$$F^{\alpha}_{\rightarrow}(O,x;m)=\sum_{m_0=0}^{\infty}\cdots\sum_{m_{x-1}=0}^{\infty}\sum_{\beta} H^{\alpha}_{L\beta}(O;m_0)p_1(\beta)\sum_{\beta} H^{\rightarrow}_{L\beta}(O;m_1)p_1(\beta)\cdots\sum_{\beta} H^{\rightarrow}_{L\beta}(O;m_{x-1})p_1(\beta)\delta_{m,m_0+\cdots m_{x-1}-x}, \tag{21}$$

where $H^{\alpha}_{L\beta}(O;t)$ refers to the left half comb. Let us define

$$Q^{\rightarrow}_{L}(\lambda)\equiv\sum_{\beta}\tilde{H}^{\rightarrow}_{L\beta}(O,\lambda)p_1(\beta), \tag{22}$$

so that for $\alpha=\rightarrow$,

$$\langle\tilde{x}(\lambda)\rangle=\frac{\tilde{P}^{\rightarrow}(O;\lambda)\sum_{x} x\lambda[Q^{\rightarrow}_{L}(\lambda)]^{x}}{\tilde{P}^{\rightarrow}(O;\lambda)\sum_{x}\lambda[Q^{\rightarrow}_{L}(\lambda)]^{x}}=\frac{\dfrac{\tilde{P}^{\rightarrow}(O;\lambda)Q^{\rightarrow}_{L}(\lambda)}{[1-Q^{\rightarrow}_{L}(\lambda)]^{2}}}{\dfrac{\tilde{P}^{\rightarrow}(O;\lambda)}{1-Q^{\rightarrow}_{L}(\lambda)}}. \tag{23}$$

Analogous relations hold for $\alpha=\leftarrow$, $\alpha=\uparrow$, and $\alpha=\downarrow$. As in the case of the random walks in an electric field, the crucial point is to know whether the quantity $Q^{\rightarrow}_{L}(\lambda)$ tends to 1 while $\lambda\rightarrow 1$, as well as whether $\tilde{P}^{\rightarrow}(O,\lambda)$ diverges in this same limit.

Let us start from $H^{\alpha}_{L\beta}(O;t)$, which can be evaluated using the translational invariance of the comb along the backbone direction. Let $I^{\alpha}_{L\beta}(O;t)$ be the probability of returning to the starting point for the first time on a left half comb. We can write the following recursion relation for the generating functions:

$$\tilde{H}^{\alpha}_{L\beta}(O;\lambda)=\delta_{\alpha,\beta}+\tilde{I}^{\alpha}_{L\uparrow}(O;\lambda)\tilde{H}^{\uparrow}_{L\beta}(O;\lambda)+\tilde{I}^{\alpha}_{L\downarrow}(O;\lambda)\tilde{H}^{\downarrow}_{L\beta}(O;\lambda)+\tilde{I}^{\alpha}_{L\rightarrow}(O;\lambda)\tilde{H}^{\rightarrow}_{L\beta}(O;\lambda). \tag{24}$$

Since

$$\tilde{I}^{\alpha}_{L\downarrow}(O;\lambda)=\frac{\lambda^2}{2}p_3(\alpha)\frac{2}{\lambda^2}(1-\sqrt{1-\lambda^2}), \tag{25}$$

we easily obtain the final expression:

$$\tilde{H}^{\alpha}_{L\beta}(O;\lambda)=\delta_{\alpha,\beta}+\lambda p_4(\alpha)\tilde{H}^{\uparrow}_{L\beta}(O;\lambda)+p_3(\alpha)(1-\sqrt{1-\lambda^2})\tilde{H}^{\downarrow}_{L\beta}(O;\lambda)+\lambda^2 p_2(\alpha)\sum_{\gamma}\tilde{H}^{\leftarrow}_{L\gamma}(\lambda)p_1(\gamma)\tilde{H}^{\rightarrow}_{L\beta}(\lambda). \tag{26}$$

Now we multiply this equation for $p_1(\beta)$ and sum over β to obtain

$$Q^{\alpha}_{L}(\lambda)=\sum_{\beta}\delta_{\alpha,\beta}p_1(\beta)+\lambda p_4(\alpha)Q^{\uparrow}_{L}(\lambda)+p_3(\alpha)(1-\sqrt{1-\lambda^2})Q^{\downarrow}_{L}(\lambda)+\lambda^2 p_2(\alpha)Q^{\leftarrow}_{L}(\lambda)Q^{\rightarrow}_{L}(\lambda). \tag{27}$$

This equation splits into a system of four equations corresponding to the four values of α. We solve the system to obtain the values of $Q^{\uparrow}_{L}(\lambda)$, $Q^{\downarrow}_{L}(\lambda)$, $Q^{\rightarrow}_{L}(\lambda)$, and $Q^{\leftarrow}_{L}(\lambda)$. In particular, in the limit $\lambda=1-\epsilon$, $\epsilon\rightarrow 0$ $Q^{\downarrow}_{L}(1-\epsilon)$ is given by

$$Q^{\downarrow}_{L}(1-\epsilon)=\frac{-4\mathcal{B}^4-2\mathcal{B}^2+\mathcal{B}+1-(3\mathcal{B}+1)|\mathcal{B}|}{-4\mathcal{B}^4+\mathcal{B}^2+2\mathcal{B}+1}+O(\sqrt{\epsilon}). \tag{28}$$

The solutions are now different depending on $\mathcal{B}$ being greater or less than zero. For $\mathcal{B}<0$, we find

$$Q^{\uparrow}_{L}(1-\epsilon)=Q^{\downarrow}_{L}(1-\epsilon)=Q^{\rightarrow}_{L}(1-\epsilon)=Q^{\leftarrow}_{L}(1-\epsilon)=1+O(\sqrt{\epsilon}), \tag{29}$$

while for $\mathcal{B}>0$ we find

$$Q^{\downarrow}_{L}(1-\epsilon)=\frac{1-5\mathcal{B}^2-4\mathcal{B}^4}{1+2\mathcal{B}+\mathcal{B}^2-4\mathcal{B}^4}+O(\sqrt{\epsilon}), \tag{30}$$

$$Q^{\uparrow}_{L}(1-\epsilon)=\frac{2\mathcal{B}+(1+\mathcal{B})Q^{\downarrow}_{L}(1-\epsilon)}{3\mathcal{B}+1}+O(\sqrt{\epsilon}), \tag{31}$$

TABLE I. Combined effect of an electric and a magnetic field.

Direction of $\mathcal{E}$	Direction of $\mathcal{B}$	Relative strength	$\langle x\rangle_R$	$\langle x\rangle_L$
$\mathcal{E}>0$	$\mathcal{B}<0$		$C_2(\mathcal{E},\mathcal{B})t^{1/2}$	$C_1(-\mathcal{E},-\mathcal{B})$
$\mathcal{E}<0$	$\mathcal{B}>0$		$C_1(\mathcal{E},\mathcal{B})$	$C_2(-\mathcal{E},-\mathcal{B})t^{1/2}$
		$\mathcal{E}>\mathcal{B}/4$	$C_2(\mathcal{E},\mathcal{B})t^{1/2}$	$C_1(-\mathcal{E},-\mathcal{B})$
$\mathcal{E}>0$	$\mathcal{B}>0$			
		$\mathcal{E}<\mathcal{B}/4$	$C_1(\mathcal{E},\mathcal{B})$	$C_2(-\mathcal{E},-\mathcal{B})t^{1/2}$
		$\lvert\mathcal{E}\rvert<\lvert\mathcal{B}\rvert/4$	$C_2(\mathcal{E},\mathcal{B})t^{1/2}$	$C_1(-\mathcal{E},-\mathcal{B})$
$\mathcal{E}<0$	$\mathcal{B}<0$			
		$\lvert\mathcal{E}\rvert>\lvert\mathcal{B}\rvert/4$	$C_1(\mathcal{E},\mathcal{B})$	$C_2(-\mathcal{E},-\mathcal{B})t^{1/2}$

$$Q_L^{\rightarrow}(1-\epsilon)=Q_L^{\uparrow}(1-\epsilon)(1+\mathcal{B})-\mathcal{B}Q_L^{\downarrow}(1-\epsilon), \quad (32)$$

$$Q_L^{\leftarrow}(1-\epsilon)=Q_L^{\uparrow}(1-\epsilon)(1-\mathcal{B})+\mathcal{B}Q_L^{\downarrow}(1-\epsilon)+O(\sqrt{\epsilon}). \quad (33)$$

This result also implies that the $Q_L^{\alpha}(\lambda)$ does not diverge as $\lambda\rightarrow 1$. For $P^{\rightarrow}(0,\lambda)$, one obtains that also this quantity is finite in the $\lambda\rightarrow 1$ limit. So we conclude that for $\mathcal{B}>0$, the leading term in the asymptotic expression of the average displacement on the right of point O has no time dependence:

$$\langle x\rangle_R\sim C_1(\mathcal{E}=0,\mathcal{B}), \quad t\rightarrow\infty, \quad (34)$$

while when $\mathcal{B}<0$ the same quantity grows as

$$\langle x\rangle_R\sim C_2(\mathcal{E}=0,\mathcal{B})t^{1/2}, \quad t\rightarrow\infty; \quad (35)$$

where

$$C_2(\mathcal{E},\mathcal{B})=\frac{\sqrt{2\pi}(4\mathcal{E}-\mathcal{B})}{1-8\mathcal{E}\mathcal{B}}, \quad (36)$$

and the explicit calculation of $C_2(\mathcal{E},\mathcal{B})$ is described in the Appendix. It is easy to show that the left average displacement can be extracted from the right one simply by changing $\mathcal{B}$ in $-\mathcal{B}$. This implies that for $\mathcal{B}>0$,

$$\langle x\rangle_L\sim C_2(\mathcal{E}=0,\mathcal{B})t^{1/2}, \quad t\rightarrow\infty; \quad (37)$$

while for $\mathcal{B}<0$,

$$\langle x\rangle_L\sim C_1(\mathcal{E}=0,\mathcal{B}), \quad t\rightarrow\infty. \quad (38)$$

V. HALL EFFECT ON A COMB

Let us now apply an electric and a magnetic field together: this will lead to different behaviors depending on the relative signs of $\mathcal{E}$ and $\mathcal{B}$. If $\mathcal{E}>0$ and $\mathcal{B}<0$, the two fields add their effects to push the walker to the right, so that

$$\langle x\rangle_R\sim C_2(\mathcal{E}=0,\mathcal{B})t^{1/2}, \quad \langle x\rangle_L\sim C_1(\mathcal{E}=0,\mathcal{B}), \quad t\rightarrow\infty. \quad (39)$$

On the contrary, if we put $\mathcal{E}<0$ and $\mathcal{B}>0$, we obtain

$$\langle x\rangle_R\sim C_1(\mathcal{E}=0,\mathcal{B}), \quad \langle x\rangle_L\sim C_2(\mathcal{E}=0,\mathcal{B})t^{1/2}, \quad t\rightarrow\infty. \quad (40)$$

The most interesting situation is when $\mathcal{E}>0$ and $\mathcal{B}>0$, or when $\mathcal{E}<0$ and $\mathcal{B}<0$, since in these cases there is a competition between the effects of the two fields, i.e., they push the walker towards opposite directions. Following the same steps described in the preceding section, we obtain two different expressions for $Q_L^{\alpha}(\lambda)$ depending on the sign of the quantity:

$$\delta\equiv(3\mathcal{B}+1-4\mathcal{E})(\mathcal{B}-4\mathcal{E}), \quad (41)$$

which is the square root of the Δ of the second-order equation we must solve to find the expression of $Q_L^{\downarrow}(\lambda)$ If $\delta<0$,

$$Q_L^{\uparrow}(1-\epsilon)=Q_L^{\downarrow}(1-\epsilon)=Q_L^{\rightarrow}(1-\epsilon)=Q_L^{\leftarrow}(1-\epsilon)=1+O(\sqrt{\epsilon}). \quad (42)$$

If $\delta>0$, we solve the second-order equation in $\mathcal{E}$ and obtain that for

$$\frac{\mathcal{B}}{4}<\mathcal{E}<\frac{1+3\mathcal{B}}{4}, \quad (43)$$

we are in the same situation of the case $\delta<0$, and the $Q_L^{\alpha}(\lambda)$ are given by Eq. (42). For

$$\mathcal{E}<\frac{\mathcal{B}}{4} \quad \vee \quad \mathcal{E}>\frac{1+3\mathcal{B}}{4}, \quad (44)$$

all the $Q_L^{\alpha}(\lambda)$ do not tend to 1 as $\lambda\rightarrow 1$. Using these equations together with the relations

$$\frac{1}{4}+\mathcal{E}-\mathcal{B}>0,$$

$$\frac{1}{4}\mathcal{E}+\mathcal{B}>0,$$

following from the positivity of the jumping probabilities, we obtain the final results that are summarized in Table I. Notice that the average left displacement has been obtained from the previous calculations simply by changing the signs of $\mathcal{E}$ and $\mathcal{B}$. The cases $\mathcal{E}=\mathcal{B}/4$ and $\mathcal{E}=(1+3\mathcal{B})/4$ correspond to the situation $\delta=0$ and describe a walker that is pushed by

$\mathcal{E}$ and $\mathcal{B}$ towards opposite directions with the same strength. As a consequence, diffusion follows the same rules of the unbiased case $\mathcal{E}=\mathcal{B}=0$ with $\langle x\rangle_R$ and $\langle x\rangle_L$ growing both as $t^{1/2}$ and with a resulting mean displacement equal to zero.

VI. SUMMARY AND DISCUSSION

The problem of the diffusion of a classical charged particle in a magnetic field has been described by random walks with one-step memory. This is a generalization of the well-known discrete time random walks problem with the additional prescription that the jumping probabilities result to be affected by the direction of the preceding step. In the particular case of the $2-d$ comb lattice, we obtained analytically the asymptotic behaviors for diffusion laws along the backbone under the simultaneous presence of an electric and a magnetic field (Hall effect). Notice that these changes are also a product of the substantial ''asymmetry'' of the comb, i.e., of the fact that the upper half comb has a different structure with respect to the lower half. If we consider a comb where the teeth are complete infinite linear chains but complete linear chains, no Hall effect can be put into evidence. The technique we developed can be extended to the whole family of the n-dimensional comb lattices [9] where we find Hall effect as in the $2-d$ comb lattice case. Moreover, the study of the effect of a magnetic field using random walk with memory applies to all the structures where the random walk problem can be analytically solved and a proper definition of the magnetic field effects is allowed.

APPENDIX

The calculation of the coefficient $C_1(\mathcal{E},\mathcal{B})$ follows from the evaluation of Eq. (23). Let us define the following functions of $\mathcal{E}$ and $\mathcal{B}$:

$$a(\mathcal{E},\mathcal{B})=-4\mathcal{B}^4+\mathcal{B}^2+2\mathcal{B}+1+8\mathcal{E}(2\mathcal{E}-\mathcal{B}-1), \quad \text{(A1)}$$

$$c(\mathcal{E},\mathcal{B})=-4\mathcal{B}^4-5\mathcal{B}^2+1+24\mathcal{E}\mathcal{B}-16\mathcal{E}^2. \quad \text{(A2)}$$

The finite part of $Q_L^{\rightarrow}(\lambda)$ in the limit $\lambda=1-\epsilon,\ \epsilon\to 0$ is

$$Q(\mathcal{E},\mathcal{B})=\frac{1}{a(\mathcal{E},\mathcal{B})}(-4\mathcal{B}^4+4\mathcal{B}^3-\mathcal{B}^2+1+8\mathcal{E}\mathcal{B}-16\mathcal{E}\mathcal{B}^2), \quad \text{(A3)}$$

while the coefficient of $\sqrt{\epsilon}$ in the expression of $Q_L^{\rightarrow}(1-\epsilon)$ is

$$\epsilon_L^{\rightarrow}(\mathcal{E},\mathcal{B})=\mathcal{B}\frac{c(\mathcal{E},\mathcal{B})}{a(\mathcal{E},\mathcal{B})}+\frac{c(\mathcal{E},\mathcal{B})}{2[a(\mathcal{E},\mathcal{B})]^2(\mathcal{B}-4\mathcal{E})(1-4\mathcal{E}+3\mathcal{B})}(-1+4\mathcal{E}-\mathcal{B}+2\mathcal{B}^2)(1-8\mathcal{E}-16\mathcal{E}^2+4\mathcal{B}-16\mathcal{E}\mathcal{B}+3\mathcal{B}^2-8\mathcal{E}\mathcal{B}^2$$
$$+32\mathcal{E}^2\mathcal{B}^2+2\mathcal{B}^3-64\mathcal{E}\mathcal{B}^3+10\mathcal{B}^4-80\mathcal{E}\mathcal{B}^4+12\mathcal{B}^5). \quad \text{(A4)}$$

As much as concerns the evaluation of $\tilde{P}^{\rightarrow}(O;\lambda)$, it can be obtained from the corresponding first time arrival generating function $\tilde{F}^{\rightarrow}(O;\lambda)$ through the relation

$$\tilde{P}^{\rightarrow}(O;\lambda)=\frac{1}{1-F^{\rightarrow}(O;\lambda)}.$$

The finite part of $\tilde{F}^{\rightarrow}(O;\lambda)$ is

$$F(\mathcal{E},\mathcal{B})=\frac{1}{2}+\left(\frac{1}{4}-\mathcal{E}\right)\frac{a(-\mathcal{E},-\mathcal{B})}{a(\mathcal{E},\mathcal{B})}Q(-\mathcal{E},-\mathcal{B})+\left(\frac{1}{4}+\mathcal{E}\right)\frac{a(-\mathcal{E},-\mathcal{B})}{a(\mathcal{E},\mathcal{B})}Q(\mathcal{E},\mathcal{B}), \quad \text{(A5)}$$

and the coefficient of $\sqrt{\epsilon}$ is

$$f(\mathcal{E},\mathcal{B})=-\left(\frac{1}{4}+\mathcal{B}\right)-\left(\frac{1}{4}-\mathcal{E}\right)\epsilon_L^{\leftarrow}(\mathcal{E},\mathcal{B})+\left(\frac{1}{4}+\mathcal{E}\right)\epsilon_L^{\leftarrow}(-\mathcal{E},-\mathcal{B}), \quad \text{(A6)}$$

where

$$\epsilon_L^{\leftarrow}(\mathcal{E},\mathcal{B})=-\mathcal{B}\frac{c(\mathcal{E},\mathcal{B})}{a(\mathcal{E},\mathcal{B})}+\frac{c(\mathcal{E},\mathcal{B})}{2[a(\mathcal{E},\mathcal{B})]^2(\mathcal{B}-4\mathcal{E})(1-4\mathcal{E}+3\mathcal{B})}(-1+4\mathcal{E}-\mathcal{B}-2\mathcal{B}^2)(1-8\mathcal{E}+16\mathcal{E}^2+4\mathcal{B}-16\mathcal{E}\mathcal{B}+3\mathcal{B}^2+8\mathcal{E}\mathcal{B}^2$$
$$-32\mathcal{E}^2\mathcal{B}^2-2\mathcal{B}^3-2\mathcal{B}^4-80\mathcal{E}\mathcal{B}^4+12\mathcal{B}^5) \quad \text{(A7)}$$

is the coefficient of $\sqrt{\epsilon}$ in the expression of $Q_L^{\leftarrow}(1-\epsilon)$. After using Tauberian theorems the final expression of $C_1(\mathcal{E},\mathcal{B})$ results to be

$$C_1(\mathcal{E},\mathcal{B})=\frac{\left(\frac{1}{1-f(\mathcal{E},\mathcal{B})}+\vec{\epsilon_L}(\mathcal{E},\mathcal{B})\right)[1-Q(\mathcal{E},\mathcal{B})]^2+2\vec{\epsilon_L}(\mathcal{E},\mathcal{B})\frac{Q(\mathcal{E},\mathcal{B})}{1-F(\mathcal{E},\mathcal{B})}}{[1-Q(\mathcal{E},\mathcal{B})]^2\left(\frac{1-Q(\mathcal{E},\mathcal{B})}{1-f(\mathcal{E},\mathcal{B})}+\frac{\vec{\epsilon_L}(\mathcal{E},\mathcal{B})}{1-F(\mathcal{E},\mathcal{B})}\right)}. \tag{A8}$$

[1] S. Havlin and D. Ben-Avraham, Adv. Phys. **51**, 187 (2002).

[2] M. Sahimi, *Applications of Percolation Theory* (Taylor & Francis, London, 1994).

[3] A.I. Onipko, Phys. Rev. B **43**, 13 528 (1991).

[4] A.S. Silva, R. Bond, F. Plouraboue, and D. Wirtz, Phys. Rev. E **54**, 5502 (1996).

[5] A.J. Breeze, Z. Schlesinger, S.A. Carter, and P.J. Brock, Phys. Rev. B **64**, 125205 (2001).

[6] W.Y. Duan, H. Yano, and Y. Tanimoto, Fractals **6**, 145 (1998).

[7] C. Aslangul, M. Barthelemy, N. Pottier, and D. Saint-James, Mod. Phys. Lett. B **9**, 601 (1995).

[8] E.W. Montroll and G.W. Weiss, J. Math. Phys. **6**, 167 (1965).

[9] D. Cassi and S. Regina, Phys. Rev. Lett. **76**, 2914 (1996); D. Cassi and S. Regina, Mod. Phys. Lett. B **6**, 1397 (1992).

[10] G. Polya, Math. Ann. **84**, 149 (1921).

[11] S. Alexander and R. Orbach, J. Phys. (France) Lett. **43**, L625 (1982).

[12] G. Giusiano, Tesi di Laurea, Università di Parma, 1999 (unpublished).