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ON PHASE TRANSITIONS ON BETHE LATTICES

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Starting from some recent rigorous results about correlation functions of statistical model on tree structures, we analyze the nature of phase transitions occurring on Bethe lattices, showing the lack of long range order and the purely geometrical origin of the thermodynamic singularities. This approach gives a very simple formula for the critical temperature for any model with compact symmetry group and immediately leads to the value 1 for the critical exponent γ. The "geometrical" critical behavior only partially coincides with the mean field solution and violates the usual scaling relations.

Since the seminal paper of Bethe[1] where Bethe lattices (BL) were introduced to approximate the local geometry of real lattices, almost all statistical models have been studied on these graphs. In fact their simple and highly symmetrical structure often allows to obtain exact results, while their infinite effective dimension leads to phase transitions similar to the mean field ones. However the results show a dramatic dependence on the way one performs the thermodynamical limit, since the number of surface sites of any finite connected sublattice of BL is of the same order of the number of internal sites. Usually one refers to BL when dealing with an infinite tree having the same coordination number $z > 2$ at each site, while the term Cayley tree (CT) is used for the finite trees whose sequence "converges" to a BL in the thermodynamic limit. It has been pointed out by several authors that if one considers from the beginning an infinite BL instead of a sequence of CT, the phase transitions occurring for Ising and Potts models are of a rather peculiar kind, since they do not exhibit long range order.[2]

In this letter we show that this result is far more general, holding for all statistical models with nearest neighbors interactions invariant under a compact transitive

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symmetry group, and we further investigate the nature of the phase transition. The proof is very simple and relies on a recent result concerning such a class of models on trees. This approach also allows to obtain a simple formula giving the critical temperature of each model.

Let us consider the model defined on a BL by the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \sigma_i \cdot \sigma_j - h \sum_i \sigma_i \,, \tag{1}$$

where the σ_i are generic scalar or vector variables with norm 1 taking values on a compact homogeneous space (for example, spin variables in an O(n) model), the dot products are invariant under the action of a compact group G (O(n) in the previous example), the first sum runs over n.n. sites, and h is an external magnetic field.

It has been shown[3] that on any tree, for $h = 0$, the two-point correlation functions $\langle \sigma_i \sigma_k \rangle$ coincide with the same functions calculated on a linear chain between two points having the same distance d_{ik} as i and k. This implies that they can be written as[4]

$$\langle \sigma_i \sigma_k \rangle = c u^{d_{ik}} \,, \tag{2}$$

where c is a model-dependent constant and u is a known model-dependent monotonically decreasing analytical function of the temperature T, with $u = 1$ at $T = 0$ and $u \to 0$ for $T \to \infty$. This immediately implies the absence of long range order, since $\langle \sigma_i \sigma_k \rangle \to 0$ for $d_{ik} \to \infty$ at any finite temperature.

This condition reduces the thermodynamics of a statistical model on any tree embeddable in a finite dimensional space to the one-dimensional case.[3] However the peculiar geometry of the BL and in particular its infinite effective dimensionality leads to nontrivial differences. In fact the number $N(r)$ of points at a distance r from a given origin growing according to a power law in the finite dimensional case here has an exponential behavior given by

$$N(r) = z(z-1)^{r-1} \tag{3}$$

for integer $r \geq 1$. This exponential dependence deeply affects the thermodynamical functions depending on large-scale geometry. In particular let us consider the zero-field susceptibility per site χ, given by

$$\chi = \sum_i \langle \sigma_O \sigma_i \rangle - \langle \sigma_O \rangle \langle \sigma_i \rangle \,, \tag{4}$$

where O is a fixed origin and the sum runs over all sites of the BL. Let us begin by considering the case where the local magnetizations are equal to 0 (this is always true at sufficiently high temperatures). Then this sum always converges when (2) holds and $N(r)$ grows slower than exponentially, as for a tree embeddable in a finite dimensional space: in this case the absence of divergences up to $T = 0$ also implies the lack of spontaneous magnetization at any finite temperature.[3] On BL, however, the exponential growth of $N(r)$ competes with the exponential decay of the correlation functions and at a given temperature leads to a divergence in χ. In fact (4) becomes

$$\chi = 1 + c \sum_{r=1}^{\infty} z(z-1)^{r-1} u^r = 1 + \frac{cuz}{1-(z-1)u(T)} \,. \tag{5}$$

This quantity converges as far as $u(z-1) < 1$, i.e. at high temperatures, while it diverges at a critical temperature T_c given by

$$u(T_c)(z-1) = 1 \,. \tag{6}$$

One can easily verify that this expression holds for all models already solved: for example, if we consider the Ising model,[5] where $u = \tanh(J/kT)$, we immediately obtain $T_c = 2J/k \ln(z/z-2)$. Due to the properties of $u(T)$, we can expand (5) for $T \to T_c$, obtaining

$$\chi \sim -\frac{cu_c z}{(z-1)u'_c}(T - T_c)^{-1} \,, \tag{7}$$

where $u_c \equiv u(T_c)$ and $u'_c = u'(T_c) < 0$. Expression (7) is the usual Curie–Weiss law appearing in mean field solution and implies $\gamma = 1$ for $T > T_c$. However the regime $T < T_c$ is different from the mean field behavior. Indeed in that case a spontaneous magnetization appears while χ is no more divergent. It is straightforward to verify that on BL this is not the case: if we suppose $\langle \sigma_i \rangle = m \neq 0$, m being the same for every i due to the homogeneity of BL, (5) would give a negative divergent χ; this implies $m = 0$ and $\chi = +\infty$ for any $T < T_c$. Notice that even relaxing the condition of a homogeneous value of $\langle \sigma_i \rangle$, as is necessary in the case of CT, we would obtain the same result considering the average of $\langle \sigma_i \rangle$ over all sites instead of m.[6]

As for the specific heat per bond, C, the result coincides with the one-dimensional case. Indeed the mean energy per bond, E, is proportional to the thermal average of the nearest neighbors correlation function, being equal to the linear chain one, and $C = dE/dT$. So C is finite for every $T > 0$. The same result can be deduced for the free energy per bond, F, that is showed to be analytical for $T > 0$.

In conclusion we showed that the only difference between BL and a linear chain or a tree embeddable in a finite dimensional space reduces to the susceptibility behavior. This difference is only due to the large-scale geometry of BL and therefore appears in a typical nonlocal thermodynamical function while all essentially local or microscopical quantities are still analytical in T.

References

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