Modern Physics Letters B, Vol. 7, Nos. 29 & 30 (1993) 1947–1950

ABSENCE OF PHASE TRANSITIONS ON TREE STRUCTURES

RAFFAELLA BURIONI*
Dipartimento di Fisica, Università di Roma "La Sapienza", Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Piazzale A. Moro 1, 00185 Roma, Italy

DAVIDE CASSI†
Dipartimento di Fisica, Università di Parma, INFM-CISM, Unità di Parma, Istituto Nazionale di Fisica Nucleare, Gruppo Collegato di Parma, Viale delle Scienze, 43100 Parma, Italy

Received 2 November 1993

We rigorously prove that the correlation functions of any statistical model having a compact transitive symmetry group and nearest-neighbor interactions on any tree structure are equal to the corresponding ones on a linear chain. The exponential decay of the latter implies the absence of long-range order on any tree. On the other hand, for trees with exponential growth such as Bethe lattices, one can show the existence of a particular kind of mean field phase transition without long-range order.

One of the most relevant results in the study of phase transitions is the discovery of the role of systems dimensionality in determining long-range order. It is known that phase transitions with spontaneous breaking of a continuous symmetry are impossible in one and two dimensions[1] while they occur in $d \geq 3$.[2] On the other hand, a discrete symmetry cannot be broken in one dimension, but long-range order occurs for $d \geq 2$.[3] Unfortunately, these fundamental results apply only to ordered systems such as crystal lattices and cannot be extended to a lot of interesting real structures that are not translationally invariant such as fractals, polymers, glasses, and generic disordered systems. Indeed in all these cases the dimensionality cannot be generalized in a unique way and in principle it is not clear if a geometrical parameter playing the same role as d does exist. However, the determination of very general geometrical features affecting phase transitions is of primary importance not only from a theoretical point of view, but also from an experimental and technological one: Up to now, for example, we still do not know in general which

*E-mail: BURIONI@ROMA1.INFN.IT, VAXROM::BURIONI
†E-mail: CASSI@PR.INFN.IT, 37993::CASSI
PACS Nos.: 75.10.Hk, 64.60.Cn, 64.60.Ak.

geometrical structures allow a disordered material to be a permanent magnet. Recently, a necessary condition for the occurrence of phase transitions in disordered systems has been found showing that a continuous symmetry cannot be broken on recursive networks, i.e. structures where random walks are recursive.[4] Since one- and two-dimensional lattices are recursive, this result is the natural generalization of the Mermin–Wagner theorem.[1] However, this theorem does not give a sufficient condition for spontaneous symmetry breaking and up to now it was an open question whether phase transitions always occur on transient networks (structures where random walks are not recursive). Moreover, an equivalent theorem concerning discrete symmetry models is not known and it is not obvious that the critical behavior of these models is affected by the same geometrical features.

In this letter, we partially answer these open questions proving a new no-go theorem concerning a wide and important class of discrete structures. Namely, we show that no long-range order is possible at any finite temperature on tree structures for models with a compact transitive symmetry group, i.e. for most realistic models describing a phase transition.

Let us begin by considering a finite tree $\mathcal{T}$ with N sites, i.e. a loopless connected graph composed of N points and $N-1$ links joining them pairwise. We call nearest neighbors (n.n.) two points joined by a link and we will consider a statistical model with n.n. interactions defined on $\mathcal{T}$. For simplicity sake we will deal with ferromagnetic interactions. The antiferromagnetic models are fully equivalent to the ferromagnetic ones on bipartite graphs such as trees. We introduce the Hamiltonian

$$H_{\mathcal{T}} = -\sum_{ij} J_{ij}\sigma_i \cdot \sigma_j \,, \tag{1}$$

where the σ_i are generic scalar or vector variables taking values on a compact homogeneous space (for example, spin variables in an $O(n)$ model), the dot products are invariant under the action of a compact group $G(O(n)$ in the previous example), and the sum runs over n.n. sites. The J_{ij} can in general be disordered couplings provided that they satisfy the boundedness conditions $0 < J_{ij} \leq J < \infty$. The main point of our proof will consist of showing that two-point correlation functions of a model described by (1) on a generic tree are equal to the corresponding ones for the same model on a linear chain. Let us begin by considering a generic point $p \in \mathcal{T}$ and define the reduced partition function

$$Z_{\mathcal{T}}(\sigma_p) = \int \prod_{i\in(\mathcal{T}-p)} d\sigma_i e^{-\beta H_{\mathcal{T}}} \,. \tag{2}$$

Due to the G-invariance of $H_{\mathcal{T}}$, it follows that $Z_{\mathcal{T}}(\sigma_p)$ is independent of both p and σ_p and is given by

$$Z_{\mathcal{T}}(\sigma_p) = \frac{\int \prod_{i\in\mathcal{T}} d\sigma_i e^{-\beta H_{\mathcal{T}}}}{\int d\sigma_p} = \frac{Z_{\mathcal{T}}}{V} \,, \tag{3}$$

where $Z_{\mathcal{T}}$ is the complete partition function on $\mathcal{T}$, and $V = \int d\sigma_p$ is finite because of the compactness condition and independence of p.

Now let us choose two generic points i and j and let $\mathcal{C}$ be the path (unique on a tree) joining them. The number r of links in $\mathcal{C}$ is the distance between i and j. Then consider the graph obtained from $\mathcal{T}$ by cutting all the links of $\mathcal{C}$. This is a disconnected tree and each one of its $r+1$ connected components contains one and only one point of $\mathcal{C}$. Therefore we can label them with these points calling them $\mathcal{T}_l$, $l \in \mathcal{C}$. Now the two-point correlation function $\langle \sigma_i \cdot \sigma_j \rangle$ is given by

$$\langle \sigma_i \cdot \sigma_j \rangle = \frac{\int \prod_{k\in\mathcal{T}} d\sigma_k \sigma_i \cdot \sigma_j e^{-\beta H_{\mathcal{T}}}}{\int \prod_{k\in\mathcal{T}} d\sigma_k e^{-\beta H_{\mathcal{T}}}} = \frac{\int \prod_{l\in\mathcal{C}} d\sigma_l Z_{\mathcal{T}_l}(\sigma_l) \sigma_i \cdot \sigma_j e^{-\beta H_{\mathcal{C}}}}{\int \prod_{l\in\mathcal{C}} d\sigma_l Z_{\mathcal{T}_l}(\sigma_l) e^{-\beta H_{\mathcal{C}}}}, \quad (4)$$

where $H_{\mathcal{C}}$ is the restriction of $H_{\mathcal{T}}$ to $\mathcal{C}$ and the $Z_{\mathcal{T}_l}(\sigma_l)$ are the reduced partition functions defined on $\mathcal{T}_l$. From (3), it follows immediately that

$$\langle \sigma_i \cdot \sigma_j \rangle = \frac{\int \prod_{l\in\mathcal{C}} d\sigma_l \sigma_i \cdot \sigma_j e^{-\beta H_{\mathcal{C}}}}{\int \prod_{l\in\mathcal{C}} d\sigma_l e^{-\beta H_{\mathcal{C}}}} \quad (5)$$

is independent of $\mathcal{T}$ and coincides with the correlation function between two points at a distance r on a linear chain. Notice that the independence of the correlation functions of $\mathcal{T}$ implies that (5) also holds in the thermodynamic limit, i.e. for infinite trees. Now, from the boundedness conditions on the J_{ij}, it follows that the correlation functions are bounded from above by the ones obtained from the same model setting all couplings equal to J.[5] But it is known that on a linear chain the latter always decays exponentially according to[6]

$$\langle \sigma_i \cdot \sigma_{i+r} \rangle = e^{-r/\xi}, \quad (6)$$

where the correlation length ξ is a model-dependent decreasing function of the temperature $T = 1/\beta k$ that diverges as $T \to 0$. This implies that, for any finite T, $\lim_{r\to\infty} \langle \sigma_i \cdot \sigma_{i+r} \rangle = \mathcal{M}^2 = 0$, where $\mathcal{M}$ is the spontaneous magnetization. Therefore, no long-range order is possible on any tree structure. Notice that the above definition of spontaneous magnetization is not in principle equivalent to the one based on the mean of all the local magnetizations used in Ref. 4, that we will call M. However, the latter is zero when the former vanishes if the considered structure is embeddable in a finite-dimensional space.[7] More precisely, if we introduce the tree growth function at point i, $N_i(r)$ giving the number of sites at distance less or equal to r from i, M vanishes if $N(r)$ is of order less than exponential. This condition is not satisfied, for example, in the case of Bethe lattices, i.e. infinite trees with constant coordination number at each site, for which $N(r)$ grows exponentially. Various models have been solved exactly on these trees, and the results always coincide with the mean field solution, showing a phase transition at finite T. This transition is indeed of a rather particular kind, since the free energy is not singular at any finite T.[8] This kind of transition can be easily understood using our result concerning correlation functions: While $\mathcal{M}$ is obviously zero, M becomes finite in the thermodynamic limit for $T < T_c$, where T_c is defined by[7]

$$\xi(T_c) = \lim_{r\to\infty} r/\log N_i(r), \tag{7}$$

$N_i(r)$ being independent of i. Moreover, all the thermodynamics can be deduced from (6) and from the explicit form of $N(r)$. By performing explicit calculations, one can easily recover the mean field behavior of physical quantities and see that this transition does not involve long-range order, but it has a pure geometrical origin due to the exponential growth of $N(r)$.[7]

In conclusion, we have shown that all compact homogeneous statistical models have no spontaneous symmetry breaking on trees.

At a first and "simple-minded" look, this result could appear obvious and without consequences. A more careful analysis shows that it has a deep meaning and strong consequences on the very general problem of phase transition on nontranslationally invariant structures: This proof represents the first no-go theorem concerning discrete symmetry models on nontranslationally invariant lattices and also extends the results of Ref. 4. Indeed a wide class of transient trees presents subexponential growth,[9] implying the vanishing of M at any finite temperature. This proves that transience is not a sufficient condition for the spontaneous breaking of a continuous symmetry.

References

1. N. D. Mermin and H. Wagner, *Phys. Rev. Lett.* **17**, 1133 (1966).
2. J. Fröhlich, B. Simon, and T. Spencer, *Comm. Math. Phys.* **50**, 79 (1976).
3. R. Peierls, *Proc. Cambridge Phyl. Soc.* **32**, 477 (1936).
4. D. Cassi, *Phys. Rev. Lett.* **68**, 3631 (1992).
5. D. G. Kelly and S. Sherman, *J. Math. Phys.* **9**, 466 (1968).
6. See, for example, C. J. Thompson, in *Phase Transition and Critical Phenomena*, Vol. 1, eds. C. Domb and M. S. Green (Academic, London, 1972) Chap. 5.
7. R. Burioni and D. Cassi, in preparation.
8. T. P. Eggarter, *Phys. Rev.* **B9**, 2989 (1974).
9. R. Burioni and D. Cassi, UPRF-93-372, Roma1/952, June 1993.