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Classical Heisenberg and spherical model on noncrystalline structures

R. Burioni[a,b], D. Cassi[b,c,*], C. Destri[a]

[a] *Dipartimento di Fisica, Università di Milano, Via Celoria 16, 20133 Milano, Italy*
[b] *INFM Unità di Parma, Viale delle Scienze, 43100 Parma, Italy*
[c] *Dipartimento di Fisica, Università di Parma, Viale delle Scienze, 43100 Parma, Italy*

Abstract

It is known that on regular lattices the spherical model is the large-n limit of classical Heisenberg O(n) models for all temperatures. Here we give a rigorous proof of the analogous result holding in the critical regime on disordered structures (representing e.g. amorphous materials, polymers, fractals). In particular, the large-n limit of critical exponents for the Heisenberg model coincides with the critical exponents of the spherical model. These can be exactly calculated and are shown to depend only on the spectral dimension $\bar{d}$ of the structure. In addition, when $\bar{d} < 2$, as it is the case for many real structures, the critical exponents of all O(n) models coincide with the corresponding ones for the spherical model.

Keywords: Critical phenomena – Heisenberg model; Disordered ferromagnets; Fractals; 1/n expansion

The study of magnetic-spin models on noncrystalline structures is an intriguing and complex problem in statistical mechanics. This is fundamentally due to the lack of translational invariance and of a natural definition for the system dimensionality. The former gives rise mainly to technical difficulties, arising from the impossibility of using such a powerful tool as Fourier transforms. The latter involves some deeper questions concerning the role of large-scale geometry in determining the critical behavior in phase transitions. Indeed, the dimension of a crystal lattice is known to encode all relevant information about long-range topology: for a given symmetry of the spin space it is the only parameter ruling the existence of finite-temperature phase transitions and affecting the critical exponents. Therefore, one of the most challenging tasks for theoretical physicists consists in finding (if any) a generalized dimension playing a similar role for noncrystalline structures and easily measurable by experiments on real systems.

Recently, these problems have been successfully addressed using graph theory techniques. A graph, i.e. a network composed of sites and links connecting nearest-neighboring sites, is the most suitable geometrical model to describe an irregular magnetic system consisting of spins coupled by exchange interactions. This approach gives very interesting results when dealing with classical ferromagnetic Heisenberg models with O(n) symmetry, due to the deep relations between their critical behavior and low-frequency vibrational spectrum.

In particular, the spectral dimension $\bar{d}$, describing the spectral density $\rho(\omega)$ of the network in the limit $\omega \to 0$ according to $\rho(\omega) \sim \omega^{\bar{d}-1}$, appears to be the right generalization of Euclidean dimension. Not only can it be easily measured by well-established experimental techniques, such as neutron scattering, but also it is known to be deeply related to phase transitions of continuous symmetry spin systems. Indeed, the possibility of spontaneous magnetization at finite temperature for ferromagnetic Heisenberg models on generic graphs depends only on $\bar{d}$ being greater than 2 [1, 2], and the critical exponents of the spherical model are known as simple functions of $\bar{d}$ only [3]. On regular lattices the spherical model has played an important role owing to its solvability in any dimension and its connection to the

* Corresponding author. Fax: +39 521 905223; e-mail: cassi@pr.infn.it.

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Table 1
Critical exponents of the spherical model on a graph of vibrational spectral dimension $\bar{d}$

	$1 \leqslant \bar{d} < 2$	$2 < \bar{d} < 4$	$\bar{d} > 4$
α	$\bar{d}/(d-2)$	$(\bar{d}-4)/(d-2)$	0
β	–	1/2	1/2
γ	$-2/(d-2)$	$2/(d-2)$	1
δ	∞	$(\bar{d}+2)/(d-2)$	3

O(n) Heisenberg models. In fact it is well known [4, 5] that for $n \to \infty$ all thermodynamical quantities of the O(n) model tend to those of the spherical model, providing the basis for the $1/n$ expansion. Here we present how this basic result can be extended to these two statistical models defined on a generic infinite graph.

We are dealing with the standard classical O(n) Heisenberg Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j, \tag{1}$$

where the sum extends to all links of the graph, $J > 0$ and $\boldsymbol{S}_i$ is an n-dimensional vector normalized by $\boldsymbol{S}_i \cdot \boldsymbol{S}_i = n$. This constraint can be reformulated in a standard way by introducing an independent Lagrange multiplier λ_i at each site, so that the partition function reads

$$Z = \int \prod_i \mathrm{d}^n S_i \, \mathrm{d}\lambda_i \exp\left[\beta J \sum_{\langle ij \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j + i \sum_i \lambda_i (\boldsymbol{S}_i \cdot \boldsymbol{S}_i - n)\right] \tag{2}$$

and the spins are now unconstrained. The integral over the spin is now Gaussian and can be calculated exactly. The integrals over λ_i are estimated in the limit $n \to \infty$ by replacing them with the integrand evaluated at $\lambda_i = \bar{\lambda}_i$, where $\bar{\lambda}_i$ satisfy the appropriate saddle-point equations. In the thermodynamic limit the number of equations is infinite, but on crystal lattices translational symmetry allows to reduce them to just a few. These coincide with the saddle-point equations obtained in the thermodynamical limit for the spherical model which, on lattice with simple elementary cell, is defined by the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \phi_i \phi_j, \tag{3}$$

where ϕ_i are real scalar spin variables with the constraint $\sum_i \phi_i^2 = N$ and N is the number of lattice sites. This provides the equivalence of the two models for any temperature. Evidently, no such reduction for the infinite set of equation is possible on a noncrystalline structure. However, an analogous reduction does take place also on a generic graph if one considers the behavior of the system asymptotically near the critical point.

Indeed, the solution of the saddle-point equations providing the configuration of the Lagrange multipliers near the critical temperature is determined by the leading singularities of the terms appearing in the equations. It can then be rigorously proven [6] that the singular behavior is unique and the saddle-point equations again coincide, in this limit, with those of the spherical model defined on the graph by the Hamiltonian Eq. (3) with the generalized spherical constraint $\sum_i z_i \phi_i^2 = N$ where z_i is the coordination number of site i. The physical explanation relies on the nature itself of a critical point. The divergence of the correlation length implies that the geometrical structure of the graph affects the thermodynamics only through large-scale averages for which the local details of the graphs are washed out regardless of their irregularities. This result, far from being purely mathematical, allows to exactly determine all critical exponents of the O(∞) Heisenberg model, since they coincide with those of the spherical model on graphs, as given in Table 1.

We stress that these critical exponents depend only on the spectral dimension $\bar{d}$. They provide the starting point for the $1/n$ expansion on a generic graph. Furthermore, in the case of those particular graphs with $\bar{d} < 2$ (and hence $T_c = 0$) which allow for an exact solution, the critical exponents exactly coincide with those of the spherical model for any n. More generally it can be proven by heuristic arguments that this is the case for all structures having $\bar{d} < 2$. This is particularly interesting, since till now, all real structures for which $\bar{d}$ has been measured turn out to have $\bar{d} < 2$. The experimental study of the behavior of such systems at low temperature should provide a good check of this result.

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