$n \to \infty$ Limit of $O(n)$ Ferromagnetic Models on Graphs

Raffaella Burioni* and Davide Cassi†

Dipartimento di Fisica, Istituto Nazionale di Fisica della Materia, Università di Parma, Viale delle Scienze, 43100 Parma, Italy

Claudio Destri‡

Dipartimento di Fisica and INFN, Università di Milano, Via Celoria, 16 20133 Milano, Italy

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Thirty years ago, H. E. Stanley showed that an $O(n)$ spin model on a lattice tends to a spherical model as $n \to \infty$. This means that at any temperature the corresponding free energies coincide. This fundamental result is no longer valid on more general discrete structures lacking in translation invariance, i.e., on graphs. However, only the singular parts of the free energies determine the critical behavior of the two statistical models. Here we show that for ferromagnetic models such singular parts still coincide even on graphs in the thermodynamic limit. This implies that the critical exponents of $O(n)$ models on graphs for $n \to \infty$ tend to the spherical ones and depend only on the graph spectral dimension.

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The study of spin models on noncrystalline structures is an intriguing and complex problem in statistical mechanics. This is fundamentally due to the lack of translational invariance and of a natural definition for the system dimensionality. The former gives rise mainly to technical difficulties, arising from the impossibility of using such a powerful tool as Fourier transforms. The latter involves some deeper questions concerning the role of large scale geometry in phase transitions on important real structures such as amorphous solids, glasses, polymers, and fractals.

Recently, these problems have been successfully addressed using graph theory. A graph, i.e., a network composed of sites and links connecting nearest-neighboring sites, is the most suitable geometrical model to describe an irregular system consisting of spins coupled by local interactions. In particular, algebraic graph theory gives very interesting results when dealing with continuous symmetry models, due to the evidence for deep relations between their critical behavior and the small eigenvalues spectrum of the Laplacian operator.

One of the most representative models in this class is the $O(n)$ model, which describes a classical n-dimensional spin vector of fixed length. It is not in general exactly solved, but a few analytic results have been obtained on lattices. For example, the lower critical dimension for spontaneous symmetry breaking is known from the Mermin-Wagner theorem and the $n \to \infty$ limit, which corresponds to the spherical model, can be exactly solved on any lattice. These results provide the basis for a qualitative understanding of the phase diagram as well as for the $1/n$ expansion of thermodynamical quantities and, in particular, of critical exponents.

As for graphs, while an extension of the Mermin-Wagner theorem to non-translation-invariant structures has been proven [1], the $n \to \infty$ result for $O(n)$ models heavily relies on the invariance properties of the lattice and therefore cannot be easily generalized. Indeed the equivalence between an infinite component model with local constraints and a model with a global constraint on the spin is lost as a consequence of the lack of symmetry. The extension of this result to a generic discrete structure would be an important step in the comprehension of continuous symmetry models and in particular of their phase diagrams and universality classes. In this Letter, we deal with such a problem.

We show that on a graph, although the free energy of $O(n)$ models in the $n \to \infty$ limit at a generic temperature is not in general equivalent to the corresponding free energy of the spherical model, the singular parts of the two free energies coincide. Therefore on a generic discrete structure the equivalence is retrieved asymptotically near the critical point and it holds for critical exponents, which can be shown to depend only on the spectral dimension of the network. This result strongly supports the possibility of a global geometrical characterization of irregular networks in the critical region, affecting all phenomena related to large scales.

The ferromagnetic classical $O(n)$ Heisenberg model is defined by the Boltzmann weight $\exp(-\beta H_n)$, where

$$H_n[\mathbf{S}] = \frac{1}{2} \sum_{\langle ij \rangle} J_{ij} (\mathbf{S}_i - \mathbf{S}_j)^2, \tag{1}$$

the sum extends to all links of a certain graph $\mathcal{G}$ with N sites, $J_{ij} > 0$ are ferromagnetic interactions which may vary from link to link, and $\mathbf{S}_i$ is an n-dimensional vector of fixed length normalized by $\mathbf{S}_i \cdot \mathbf{S}_i = n$. The free energy per component is defined by

$$f_n(\beta) = -\frac{1}{Nn} \frac{1}{\beta} \log Z, \tag{2}$$

where the partition function reads

$$Z = \int \prod_i \delta(\mathbf{S}_i \cdot \mathbf{S}_i - n)\, d\mathbf{S}_i\, e^{-\beta H}. \tag{3}$$

We shall assume the existence of the thermodynamic limit $N \to \infty$ which turns $\mathcal{G}$ into an infinite graph within a certain class to be better specified in the sequel.

The classical fundamental result originally due to [2] applies when $\mathcal{G}$ is a regular lattice (e.g., a hypercubic lattice); it establishes a rigorous relation between $O(n)$ models in the $n \to \infty$ limit and the spherical model. The latter is defined, on the same graph $\mathcal{G}$ of the $O(n)$ model, by the Boltzmann weight $e^{-\beta H^S}$ with Gaussian Hamiltonian

$$H^S = \frac{1}{2} \sum_{\langle ij \rangle} J_{ij} (\phi_i - \phi_j)^2 \tag{4}$$

and the spherical constraint $\sum_i \phi_i^2 = N$, where ϕ_i are real scalar variables. In the thermodynamic limit $N \to \infty$ we are allowed to replace (4) by

$$H^S = \frac{1}{2} \sum_{\langle ij \rangle} J_{ij} (\phi_i - \phi_j)^2 + \frac{1}{2} m^2 \sum_i (\phi_i^2 - 1), \tag{5}$$

where the "mass" parameter m^2 is fixed to be a precise function of temperature i.e., $m^2 = \mu(\beta)$, by the spherical constraint "on the average"

$$\lim_{N \to \infty} \frac{1}{N} \sum_i \langle \phi_i^2 \rangle = 1 \,. \tag{6}$$

Since H^S is Gaussian, one finds

$$f^S(m^2) = - \lim_{N \to \infty} \frac{1}{N\beta} \log Z^S = \frac{1}{2\beta} \log \det (L + m^2) - \frac{m^2}{2}, \tag{7}$$

$$\langle \phi_i \phi_j \rangle = \frac{1}{\beta} (L + m^2)^{-1}_{ij}$$

where $L_{ij} = J_i \delta_{ij} - J_{ij}$, with $J_i = \sum_j J_{ij}$, is a discrete Laplacian operator on $\mathcal{G}$. At this stage it is convenient to better specify the class of infinite graphs and couplings with which we are concerned: we shall assume that $\mathcal{G}$ can be naturally embedded in a finite dimensional Euclidean space and that J_i is uniformly bounded over $\mathcal{G}$. For our class of graphs a generic quantity q_i, related to a single point i, can be averaged in a unique way over $\mathcal{G}$ by $[q]_{\mathcal{G}} = \lim_{N \to \infty} 1/N \sum_i q_i$. The measure $|\mathcal{G}'|$ of a subgraph $\mathcal{G}'$ is given by the average value of its characteristic function $\chi_i^{\mathcal{G}'}$, defined by $\chi_i^{\mathcal{G}'} = 1$ if $i \in \mathcal{G}'$ and $\chi_i = 0$ otherwise: $|\mathcal{G}'| = [\chi]_{\mathcal{G}}$. The asymptotic behavior of the model in the massless limit $m^2 \to 0$ is related to the spectrum of the Laplacian operator at low eigenvalues [3]. In particular, we have the following singular behaviors as $m^2 \to 0$:

$$\text{sing}[(L + m^2)^{-1}]_{\mathcal{G}} = \text{sing} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (L + m^2)^{-1}_{ii} = C' (m^2)^{(\bar{d}/2)-1}, \tag{8}$$

which define the spectral dimension $\bar{d}$ of the graph. Together with Eqs. (6) (which now reads $[\langle \phi^2 \rangle]_{\mathcal{G}} = 1$) and (7), Eq. (8) determines the asymptotic form of the function $m^2 \equiv \mu(\beta)$, that is $\mu(\beta) \sim \beta^{2/(\bar{d}-2)}$, $\beta \to \infty$ for $\bar{d} < 2$, $\mu(\beta) \sim (\beta_c - \beta)^{2/(\bar{d}-2)}$, $\beta \to \beta_c^-$ for $2 < \bar{d} < 4$, and $\mu(\beta) \sim \beta_c - \beta$ for $\bar{d} > 4$, where $\beta_c = [L^{-1}]_{\mathcal{G}}$. In turn this implies that the free energy of the spherical model [see Eq. (7)] has the asymptotic form near the critical point

$$f^S(\mu(\beta)) \simeq f^S(0) + O(\mu(\beta)). \tag{9}$$

The infrared singularity of $[(L + m^2)^{-1}]_{\mathcal{G}}$ is determined by the large scale topology of the graph $\mathcal{G}$. For most infinite graphs, if $\mathcal{G}'$ is a subgraph of $\mathcal{G}$ with positive measure, $|\mathcal{G}'| > 0$, the infrared singularities of $[(L + m^2)^1]_{\mathcal{G}}$ and $[(L + m^2)^1]_{\mathcal{G}'}$ coincide and are described by the same spectral dimension $\bar{d}$ and the same coefficient C' of the singular part, i.e., $\text{sing}[(L + m^2)^{-1}]_{\mathcal{G}} = \text{sing}[(L + m^2)^{-1}]_{\mathcal{G}'}$. This is the case for all fractals (such as Sierpinski gasket and carpet, t fractals, and so on), bundled graphs (comb lattices, brushes,...), and many other networks (e.g., NT_D [4]), which we will call "pure graphs." The result is obtained [5] starting from the diagonal terms $\sum_k (L + m^2)_{ik} \langle \phi_k \phi_i \rangle = 1$ of the Schwinger-Dyson equation, taking its averages over $\mathcal{G}'$ and over $\mathcal{G}$ and following steps very close to those in [6]. This proof does not hold for a very peculiar class of macroscopically inhomogeneous networks, obtained, for example, by sticking together two or more pure graphs (factors) with different spectral dimensions by a zero-measure set of links. In these cases, which we will call "mixed graphs," by using the random walk representation of $[(L + m^2)^{-1}]_{ii}$ [6], one can show [5] that the Gaussian free energy for the whole graph is simply the weighted sum of the Gaussian free energies of the subgraphs. Therefore one can treat the factors as independent. This reduces the study of mixed graphs to the study of a finite number of pure graphs. Therefore, in the following we will restrict to the pure case.

Let us now come back to the equivalence between the $O(\infty)$ model and the spherical one. The proof of the classical result mentioned above deeply relies on the translation invariance of lattices with simple elementary cells whenever the couplings J_{ij} are constant over the lattice. It holds in the thermodynamic limit and uses a saddle point technique [2] that cannot be generalized to a generic discrete structure. Here, we will follow a more general and flexible approach to the infinite component limit. In fact, it can be proven that, provided $0 < \epsilon < J_{ij} < J_M < \infty$ (bounded ferromagnetic couplings) [7]:

$$-\frac{1}{2n} \log[(m^2 + J_M)] \leq f_n - f^S(m^2) \leq K_1 \frac{1}{N} \sum_i \langle \phi_i^2 - 1 \rangle^2 + \frac{K_2}{n} \tag{10}$$

where the constants K_1 and K_2 do not depend on m^2, N, and n, f_n is the free energy per component for the $O(n)$

model defined by (2), $f^S(m^2)$ is that defined in Eq. (7), and the average in the right has Boltzmann weight $e^{-\beta H^S}$, with the Hamiltonian (5) with generic m^2, i.e., where m^2 is not yet fixed by the average spherical constraint (6). Notice that inequality (10) contains as a particular case the hypercubic lattice result. Indeed on a translation invariant structure $\langle \phi_i^2 \rangle$ does not vary from site to site and the spherical constraint (6), corresponding to $m^2 = \mu(\beta)$, allows the right hand side of (10) to vanish for $n \to \infty$, implying the coincidence of the two free energies.

In the case of a generic discrete structure $\langle \phi_i^2 \rangle$ does change from site to site and one would in general need site-dependent squared masses m_i^2 to enforce $\langle \phi_i^2 \rangle = 1$ for every i and obtain an analogous coincidence of the two free energies in the infinite components limit. Thus in the thermodynamic limit one would have to solve an infinite number of equations which could not be reduced to a single one due to the lack of translation invariance, forbidding the equivalence with the spherical model where only the *average* constraint (6) holds and is solved through a *single* global parametrization $m^2 = \mu(\beta)$.

The key point of our approach is that we are interested in the critical properties of the two models. Therefore we will require from the beginning only the coincidence of the *singular parts* of the two free energies. This requirement has two main consequences: the constraint equations are replaced by "constraint inequalities" and, above all, the original models can be replaced by modified models with the same singular parts of the free energies, therefore belonging to the same universality classes. Then we will show that a particular global choice $m^2 = \overline{\mu}(\beta)$ corresponding to the solution of the constraint equation for a rescaled spherical model on $\mathcal{G}$ does satisfy the constraint inequalities and therefore gives the critical behavior of corresponding rescaled $O(n)$ model in the $n \to \infty$ limit. Finally, the rescaled models will be shown to belong to the same universality class of the original ones.

The first step of our proof consists of obtaining the constraint inequalities.

Let us consider the $n \to \infty$ limit of Eq. (10), which we write as

$$f^S(m^2) \leq f_\infty \leq f^S(m^2) + K_1[\langle \phi^2 - 1 \rangle^2]_{\mathcal{G}}, \tag{11}$$

and suppose we could show that

$$[\langle \phi^2 - 1 \rangle^2]_{\mathcal{G}} \leq o(\mu(\beta)) \tag{12}$$

near the critical point of the spherical model. Then comparing with the asymptotic form (9) of $f^S(\mu(\beta))$ itself would immediately prove that the two free energies have both the same value at the critical point *and* the same singular parts near it.

Now, for the class of infinite graphs under consideration (pure graphs) [5], the inequality (12) follows from the infinite set of linearized inequalities

$$|[\langle \phi^2 - 1 \rangle_\mu]_{\mathcal{G}'}| \leq o(\mu(\beta)), \tag{13}$$

where the average is taken over every subgraph $\mathcal{G}'$ with $|\mathcal{G}'| > 0$. We shall call (13) the "constraint inequalities" by analogy with the usual approach.

As a matter of fact, it is in general impossible that inequalities (13) are verified by the solution $\mu(\beta)$ of the global constraint for the standard spherical model on $\mathcal{G}$. However, we prove the following: (i) It is possible to find a modified set of couplings $\{J'_{ij}\}$ and site-dependent masses $\{m_i'^2\}$, both functions of m^2, which define two modified models on $\mathcal{G}$ with Hamiltonians H'^S and H'_n such that the corresponding constraint inequalities (13) are satisfied by the solution of the (modified) spherical model on $\mathcal{G}$, namely $m^2 = \overline{\mu}(\beta)$, with $\overline{\mu}(\beta) = O(\mu(\beta))$ near criticality. Therefore, from the inequality (11), the two corresponding free energies $f'_\infty(\beta)$ and $f'^S(\overline{\mu}(\beta))$ have the same singular part. (ii) The rescalings $\{J_{ij}\} \to \{J'_{ij}\}$ and $\{m_i^2\} \to \{m_i'^2\}$ do not affect the singular parts of the free energies for both the $O(n)$ and the spherical models and therefore such modified models belong to the same universality classes as the original ones.

To prove the first point let us introduce the set of couplings and masses,

$$\begin{aligned} J'_{ij} &= J_{ij}\sqrt{[a_i + b_i(m^2)][a_j + b_j(m^2)]}, \\ m_i'^2 &= m^2[a_i + b_i(m^2)], \end{aligned} \tag{14}$$

where $0 < \epsilon < a_i < A < \infty$ and the $b_i(m^2)$ are functions to be determined which vanish for $m^2 \to 0$. Such a rescaling can be seen as a bounded rescaling of the scalar spin variables $\phi_i \to \phi'_i = \phi_i\sqrt{[a_i + b_i(m^2)]}$ with the original Hamiltonian (5). Therefore, the inequalities (13) for the modified model H'^S can be written in terms of the ϕ'_i. Now setting $m^2 = \mu(\beta)$ the constants a_i and the functions $b_i(\mu)$ can always be chosen self-consistently to satisfy the inequalities to $o(\mu(\beta))$. This can be done at the same time for every subgraph $\mathcal{G}'$ with $|\mathcal{G}'| > 0$, as required by Eq. (13), because the leading singularity of $[\langle \phi^2 \rangle]_{\mathcal{G}'}$, including its coefficient, is universal, that is equal to that of $[\langle \phi^2 \rangle]_{\mathcal{G}}$.

Then for $n \to \infty$ the $O(n)$ model with coupling J'_{ij} and the spherical model, defined by the same set of J'_{ij}, satisfy the constraint inequalities. Therefore, their free energies have the same singular part and the two models have the same critical properties.

Now, let us consider the universality classes of these models. The critical behavior of the spherical model on a graph can be expressed in term of its spectral dimension $\bar{d}$, defined by (8). This has been shown to be invariant under bounded rescaling of couplings J_{ij} [8] such as the ones corresponding to the set J'_{ij}. Therefore, the two spherical models described by the J_{ij} and J'_{ij} belong to the same universality class. On the other hand, by the generalized Griffith inequalities [9] the correlation functions of the $O(n)$ model corresponding H'_n has the same critical behavior of the correlation function of the model H_n. Then they belong to the same universality classes. This completes our proof.

Our result on the equivalence of critical regimes for the spherical and $O(\infty)$ models on graphs provides a key step for the comprehension of phase transitions on general networks. The spherical critical exponents, which are exactly known and depend on the spectral dimension of the graph, are the starting point for the determination of the corresponding ones for $O(n)$ models, via the $1/n$ expansion. This expansion is a fundamental tool to analyze the existence of geometrical universality classes on graphs: indeed if its coefficients can be shown to depend only on the spectral dimension $\bar{d}$, then their existence is proved.

*Email address: burioni@pr.infn.it
†Email address: cassi@pr.infn.it
‡Email address: destri@mi.infn.it

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