Lee–Yang zeros and the Ising model on the Sierpinski gasket

Raffaella Burioni, Davide Cassi and Luca Donetti

Istituto Nazionale di Fisica della Materia, Unità di Parma, Dipartimento di Fisica, Università di Parma, Parco Area delle Scienze 7a, 43100 Parma, Italy

E-mail: `raffaella.burioni@pr.infn.it` and `davide.cassi@pr.infn.it` and `donetti@prix7.fis.unipr.it`

Abstract. We study the distribution of the complex temperature zeros for the partition function of the Ising model on a Sierpinski gasket using an exact recursive relation. Although the zeros arrange on a curve pinching the real axis at $T = 0$ in the thermodynamic limit, their density vanishes asymptotically along the curve approaching the origin. This phenomenon explains the coincidence of the low-temperature regime on the Sierpinski gasket and on the linear chain.

1. Introduction

The understanding of phase transitions on non-crystalline structures has been recently improved by exact results connecting general geometrical features of networks to the existence of spontaneous symmetry breaking [1–4]. The situation is more complex when dealing with the critical behaviour of model systems. For continuous symmetry models the singularity of the free energy, which determines the critical behaviour, appears to be related to the infrared spectrum of the Laplacian operator on the network [5, 6], while this is not the case for discrete symmetries (e.g. the Ising model). There, all known results suggest that the link between critical behaviour and geometry should involve some other topological features [1, 2]. An interesting result concerns the Sierpinski gasket, a typical and widely studied fractal, where the Ising model is exactly solved. On this structure, although continuous symmetry models exhibit a power law behaviour for $T \to 0$, the Ising model has an exponential low-temperature behaviour which coincides with that found on the linear chain [1, 2]. To analyse the critical regime from an *ab initio* point of view, an interesting picture is provided by the study of the singularities of thermodynamic potentials. In 1952 Lee and Yang, in two famous papers [7, 8] first proposed their fundamental approach to phase transitions, consisting in studying the zeros of the partition function of a statistical system, considered as a function of a complex parameter. The partition function on a finite volume is a polynomial in complex activity or fugacity, so that the complete knowledge of the zeros' distribution is equivalent to the knowledge of the partition function itself and all thermodynamic quantities can be obtained from it. On a finite volume there are no real zeros, the coefficient being all real and positive. However, in the thermodynamic limit the zeros can pinch the real axis (the region of physical interest) producing a singularity in the free energy (or grand-canonical potential) [9]. The pinching points are phase transitions points on the parameter axis and the zeros distribution in their neighbourhood can be connected with the critical properties of the system [10].

Unfortunately, a complete knowledge of the zeros is very difficult to obtain except for a few exactly solvable cases. General theorems hold for the zeros' distribution in the complex magnetic field plane in a class of ferromagnetic lattice systems, including the Ising model [8, 11]. On the other hand, very little is known rigorously about the behaviour of the zeros of the partition function in the complex temperature plane, the so-called Fisher zeros [12]. In general, it is not clear if Fisher zeros arrange on smooth curves even if this is the case in some exactly solvable models. For the Ising model on regular two-dimensional lattices [12, 13], the Fisher zeros arrange on curves that cross the positive real axis at the transition point. In the one-dimensional case only two zeros (with infinite multiplicity) are found and these have a non-zero imaginary part, so that there is no singular point for the free energy.

For statistical models defined on nonperiodic discrete structures, Fisher zeros show some peculiar features, making the analysis of their density and location extremely subtle. In particular, on some hierarchical lattices (i.e. q-potts model on diamond hierarchical lattices [14]) the zeros have been shown to form a fractal set (Julia sets). In this case, while the general approach for identifying the singularity points and the critical behaviour still holds, the widely used arguments concerning scaling of singularities and zeros density with the volume must be handled carefully, as will be shown in the following.

In this paper we will study the Ising model on the Sierpinski gasket, obtaining a recursive relation for the partition function, from which the zeros of the nth stage gasket can be obtained from those of the $(n-1)$th. The distribution we obtain is fractal and pinches the real axis at $T = 0$, so that a singular point with a power law critical behaviour could be expected. However, since the zeros' density is found out to vanish exponentially in the neighbourhood of $T = 0$, these zeros do not produce any singularity of the free energy: although the zeros pinch the real axis the 'critical behaviour' is the same as the one-dimensional case.

2. Ising model on the Sierpinski gasket

The Sierpinski gasket is a fractal graph which can be built recursively with the following procedure: the initial stage ($\mathcal{G}_0$) is a triangle (three sites with three edges) and the nth stage ($\mathcal{G}_n$) is obtained joining three $\mathcal{G}_{n-1}$ at their external corners, to form a bigger triangle (figure 1). In this way $\mathcal{G}_n$ has $\frac{3}{2}(3^n - 1)$ sites, 3^{n+1} edges and its side contains 2^n edges.

The gasket is obtained as the limit for $n \to \infty$ of this procedure.

The Ising model on the gasket is defined associating the spin variable $\sigma_i = \pm 1$ to every site i of the graph, and considering a nearest-neighbours interaction between points joined by an edge (link). The Hamiltonian is, therefore,

$$E = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \tag{1}$$

where the sum runs over the couples of sites joined by a link and J is a positive constant (ferromagnetic coupling).

3. Recursive relation for the partition function

For $\mathcal{G}_0$ the partition function

$$Z = \sum_{\{\sigma_i\}} e^{-\beta E} \tag{2}$$

can be seen as a sum of the elements of the rank-three tensor M_0

$$Z_0 = \sum_{\sigma_1, \sigma_2, \sigma_3 = \pm 1} M_0^{\sigma_1 \sigma_2 \sigma_3} \tag{3}$$
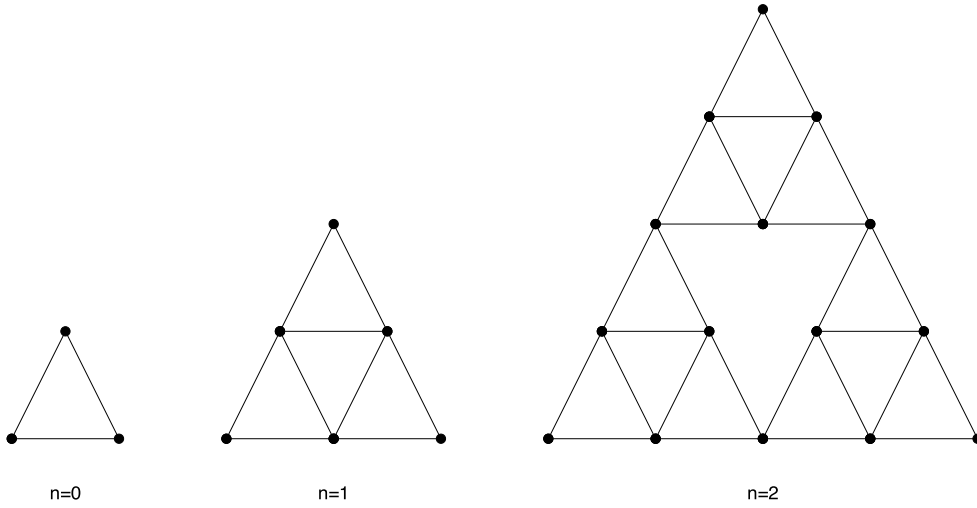
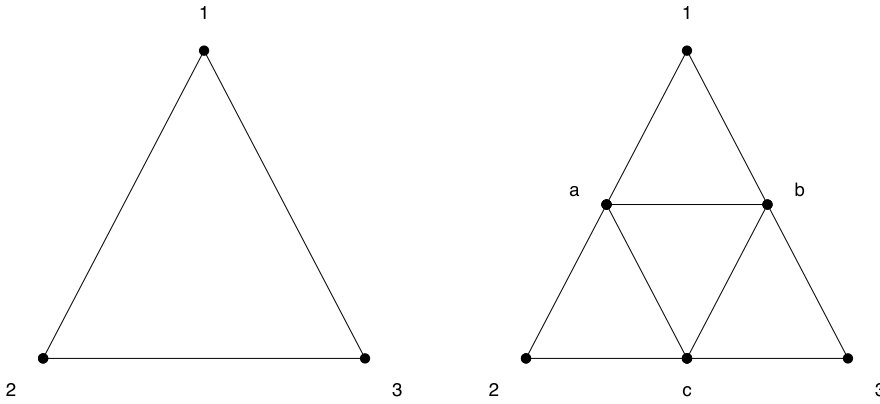**Figure 1.** First iterations of gasket's construction.

Figure 2. Labelling of sites used for $\mathcal{G}_0$ and $\mathcal{G}_1$.

where

$$M_0^{\sigma_1 \sigma_2 \sigma_3} = \exp[-\beta E(\sigma_1, \sigma_2, \sigma_3)] \tag{4}$$

$M_0^{\sigma_1 \sigma_2 \sigma_3}$ can take only two values because there are only two classes of spin configurations with different energy:

$$\begin{aligned} M_0^{\sigma \sigma \sigma} &= \mathrm{e}^{3\beta J} = y^3 \\ M_0^{\sigma \sigma (-\sigma)} &= M_0^{\sigma (-\sigma) \sigma} M_0^{(-\sigma) \sigma \sigma} = \mathrm{e}^{-\beta J} = y^{-1} \end{aligned} \tag{5}$$

where $y = \mathrm{e}^{\beta J}$. In terms of y the partition function is

$$Z_0 = 2y^3 + 6y^{-1}. \tag{6}$$

For $\mathcal{G}_1$ the partition function can be expressed in the same way separating the sum over the states of the inner sites using the tensor M_1 whose indices correspond to the spins on the external vertices (figure 2):

$$M_1^{\sigma_1 \sigma_2 \sigma_3} = \sum_{\sigma_a, \sigma_b, \sigma_c = \pm 1} \exp[-\beta E(\sigma_i)] = \sum_{\sigma_a, \sigma_b, \sigma_c = \pm 1} M_0^{\sigma_1 \sigma_a \sigma_b} M_0^{\sigma_a \sigma_2 \sigma_c} M_0^{\sigma_b \sigma_c \sigma_3}. \tag{7}$$

Now M_1 has the same structure as M_0 since the possible values are

$$M_1^{\sigma\sigma\sigma} = 4y^{-3} + 3y + y^9$$
$$M_1^{\sigma\sigma(-\sigma)} = M_1^{\sigma(-\sigma)\sigma} M_1^{(-\sigma)\sigma\sigma} = 3y^{-3} + 4y + y^5. \tag{8}$$

One can obtain M_1 from M_0 simply by a transformation mapping y^3 in $4y^{-3} + 3y + y^9$ and y^{-1} in $3y^{-3} + 4y + y^5$. This is done by the substitution

$$y \to f(y) = \left(\frac{y^8 - y^4 + 4}{y^4 + 3}\right)^{\frac{1}{4}} \tag{9}$$

followed by the multiplication by

$$c(y) = \frac{y^4 + 1}{y^3}[(y^4 + 3)^3(y^8 - y^4 + 4)]^{\frac{1}{4}}. \tag{10}$$

The transformation also gives the new partition function

$$Z_1(y) = Z_0(f(y)) c(y). \tag{11}$$

Following the same argument one can obtain for the $(n + 1)$th stage of the gasket $\mathcal{G}_{n+1}$:

$$Z_{n+1}(y) = Z_n(f(y)) \cdot [c(y)]^{3^n}. \tag{12}$$

Using this recursion relation we get

$$Z_n(y) = \frac{2}{y^{3^n}} P_n(y^4) \tag{13}$$

where $P_n(t)$ is a polynomial in t of degree 3^n in which the coefficient of t^{3^n} is 1; for $n = 0$ one has $P_0(t) = t + 3$ while the general case $n > 0$ can be proven by induction.

4. Zeros of the partition function

Introducing the variable $x = y^4$ the transformation (9), (10) is given by

$$x \to \tilde{f}(x) = \frac{x^2 - x + 4}{x + 3}$$
$$\tilde{c}(x) = (x + 1)x^{-\frac{3}{4}}\left[(x + 3)^3(x^2 - x + 4)\right]^{\frac{1}{4}}. \tag{14}$$

Denoting by x_n^i the zeros of $P_n(x)$ the partition function reads

$$Z_n = \frac{2}{x^{3^n/4}} \prod_{i=1}^{3^n}(x - x_n^i) \tag{15}$$

and using the recurrence one finds

$$2x^{-\frac{3^{n+1}}{4}} \prod_{i=1}^{3^{n+1}}(x - x_{n+1}^i) = 2x^{-\frac{3^{n+1}}{4}} \prod_{i=1}^{3^n}\{[(x^2 - x + 4) - x_n^i(x + 3)](x + 1)\}. \tag{16}$$

This equation shows that for every root x_n^i of Z_n, Z_{n+1} has the root $x = -1$ and the two solutions of

$$x_n^i = \tilde{f}(x_{n+1}^i) \tag{17}$$

namely the preimages of x_n^i by the transformation $\tilde{f}$.

Starting from $x_0^1 = -3$ one obtains all the zeros of the partition function for the nth stage gasket as shown in table 1, where $h(x)$ denotes the set of the preimages of x (and $h^k(x)$ is a set of 2^k zeros). From this table one can see that the preimages of -1 by the jth iterate of $\tilde{f}$ appear

Table 1. Zeros of partition function in y.

n	Zeros				
0	-3				
1	-1	$h(-3)$			
2	-1	$h(-1)$	$h(h(-3))$		
3	-1	$h(-1)$	$h(h(-1))$	$h(h(h(-3)))$	
n	-1	$h(-1)$	$\ldots$	$h^{n-1}(-1)$	$h^n(-3)$
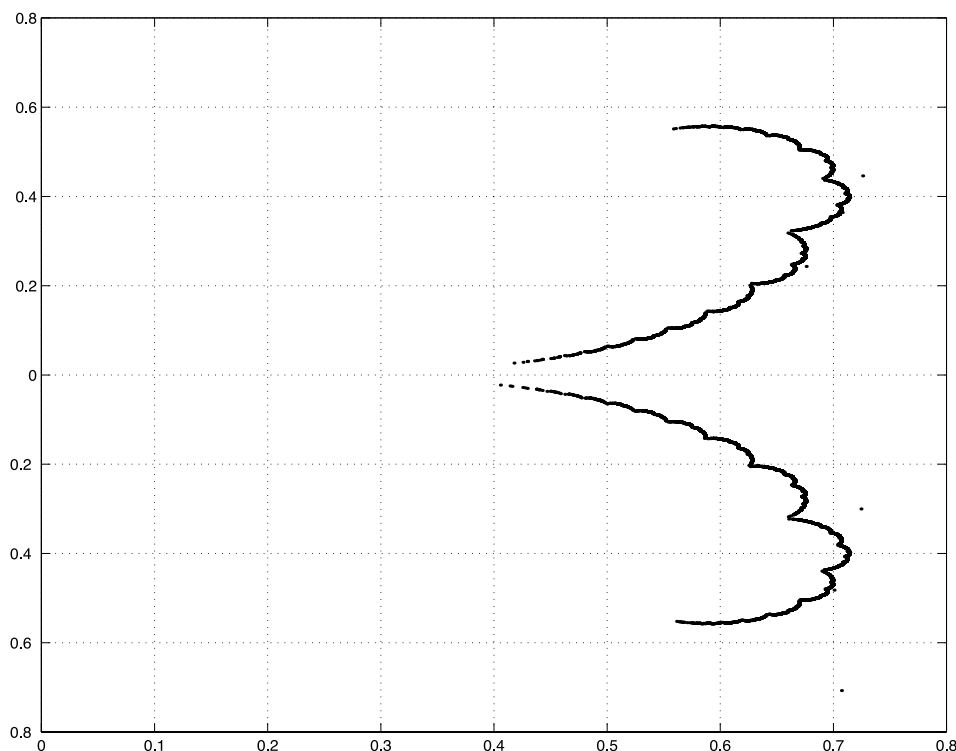
Figure 3. 20 000 zeros obtained by the random method (in the plane of $t = e^{-\beta J}$).

at the $(j-1)$th stage and are zeros of all the following stages, while the preimages of -3 are 'temporary' zeros. It is also possible to find the multiplicity of these zeros: in fact since every root generates the root -1 at the next stage, this value appears 3^{n-1} times among the zeros of nth stage, the roots $h(-1)$ appear as many times as -1 in the previous stage (their multiplicity is 3^{n-2}) and, in general, the multiplicity of the roots belonging to $h^j(-1)$ is 3^{n-j-1}.

In this way one can, in principle, calculate all the zeros of the partition function at any stage. In practice this is possible only for small n because of their exponential growth. An alternative approach [14] is to start from a root x_0 (for example '-1' whose preimages are 'permanent') then choose at random one of its two preimages by the transformation $\tilde{f}$ (denoted by x_1), then choose one of the preimages of x_1 and so on; the set of points obtained in this way is a representative of the set of all roots and has the advantage of containing zeros relative to large n.
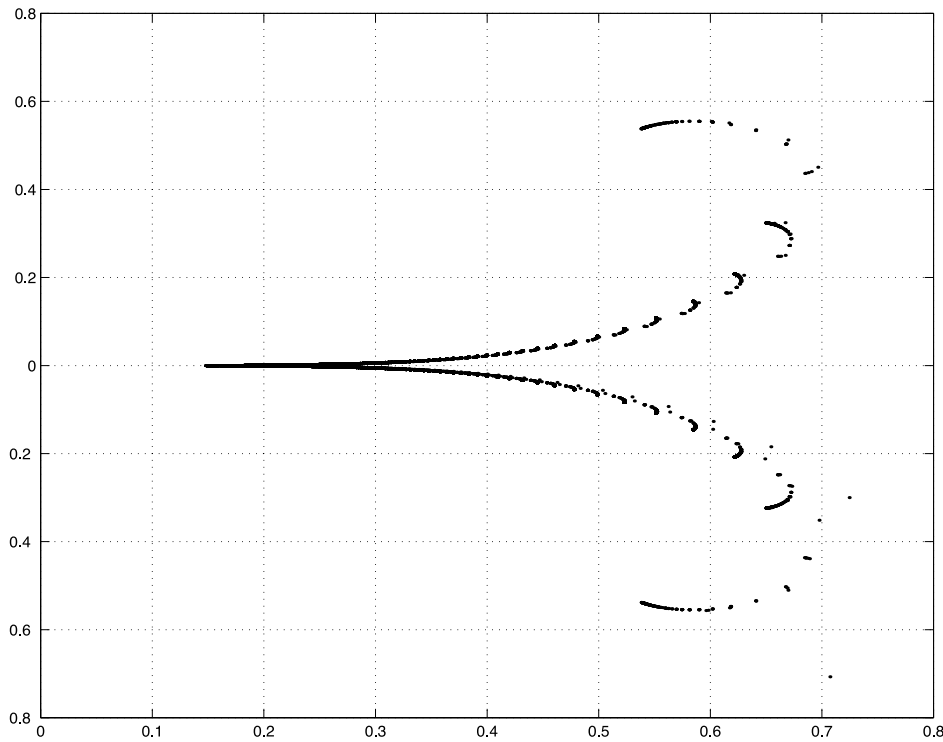
Figure 4. 20 000 zeros obtained with probability 0.98 of choosing h_2 (in the t plane).

By plotting in the complex temperature plane the roots obtained with both methods it can be seen that very few of them fall in the neighbourhood of the real axis and no information can be obtained about the critical behaviour (see figure 3 where the zeros are plotted in the plane of the variable $t = \mathrm{e}^{-\beta J}$).

A good technique to obtain more zeros near the real axis consists in changing the choice probability of the two preimages [14]; the two solutions of $x = \tilde{f}(x')$ are

$$x' = h_1(x) = \frac{1 + x - \sqrt{-15 + 14\,x + x^2}}{2} \tag{18}$$

$$x' = h_2(x) = \frac{1 + x + \sqrt{-15 + 14\,x + x^2}}{2} \tag{19}$$

and one can see that the repeated application of h_2 gives a sequence of points approaching the real axis. The set of roots obtained by increasing the probability of choosing the second preimage is not a representative set af all roots (it does not show their density, not even approximately) but gives us a chance to observe their behaviour in the interesting area (see figure 4).

A plot of these roots in the plane of $w = \mathrm{e}^{-4\beta J}$ ($T = 0$ corresponds to $w = 0$) with a log–log scale (figure 5) shows that the real and imaginary part are related by a power law: the curve can intersect the real axis in the thermodynamic limit only at the origin.

Analytically, one can verify this power behaviour by studying the transformation of the variable $w = \mathrm{e}^{-4\beta J} = x^{-1}$:

$$g(w) = \left. \frac{1}{f(x)} \right|_{x=\frac{1}{w}} = \frac{w(3w+1)}{4w^2 - w + 1}. \tag{20}$$
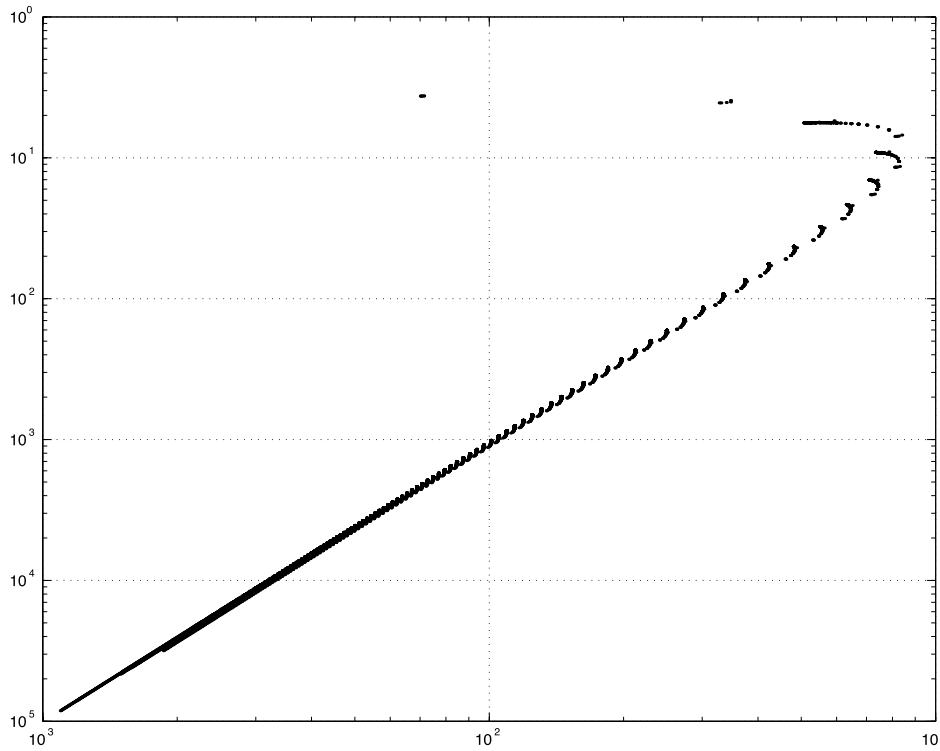
Figure 5. Log–log plot of zeros (in the variable $w = e^{-4\beta J}$).

Assuming that $\mathrm{Im}\,(w) = A\mathrm{Re}\,(w)^b$, that is

$$w = \xi + iA\xi^b \tag{21}$$

and inserting (21) in (20) one obtains

$$\mathrm{Im}\,(g(\xi + iA\xi^b)) = A\xi^b(1 + 8\xi + \mathrm{O}(\xi^{\min\{3,2b-1\}})) \tag{22}$$

and

$$A(\mathrm{Re}\,(g(\xi + iA\xi^b)))^b = A\xi^b(1 + 4b\xi\mathrm{O}(\xi^{\min\{3,2b-1\}})). \tag{23}$$

Choosing $b = 2$, one sees that the curve $w = \xi + iA\xi^2$ is 'conserved' by transformation g except for higher order terms in ξ.

5. Density of zeros

We have seen that the zeros pinch the real axis only at $T = 0$ and we proceed by studying their density in the neighbourhood of this point to establish the critical behaviour; it is important to see whether this density (which is quite small, as we have seen) goes to zero or remains finite. A numerical estimate can be obtained by simply counting the zeros (with their multiplicity).

This has been done in two ways:

- considering only the zeros in the neighbourhood of the real axis and grouping them with regard to their real part (one-dimensional density);
- dividing the complex plane in equal rectangles and counting the zeros contained (two-dimensional density).
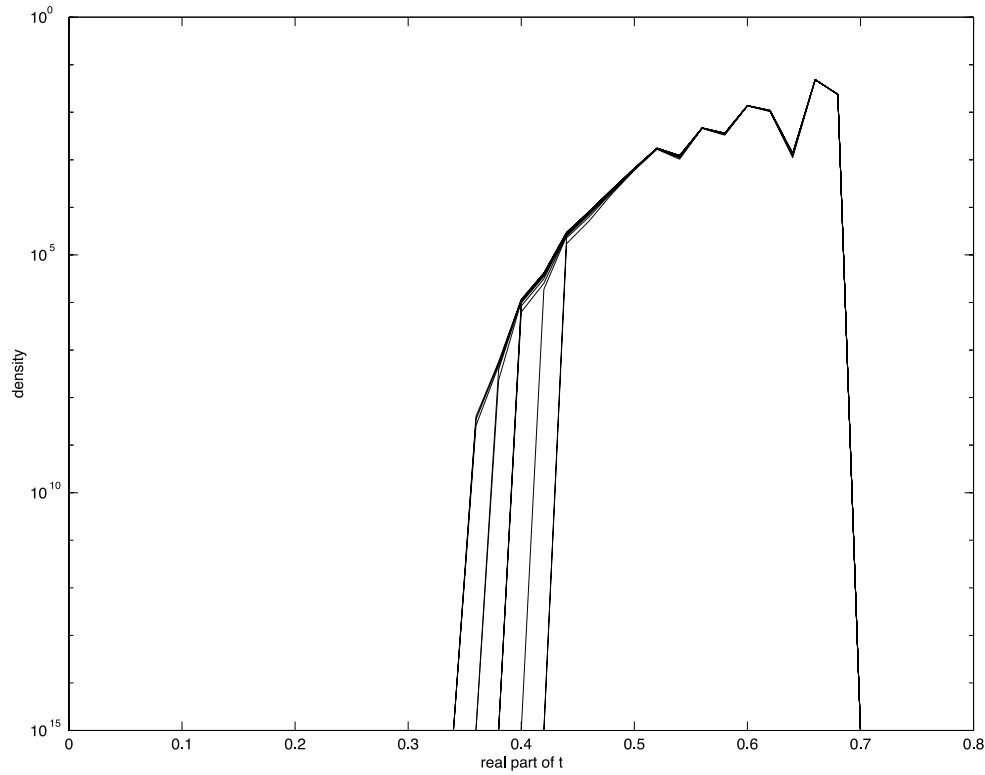
Figure 6. Density of zeros versus their real part in the t plane (I method).

In the first case one obtains plots like figure 6 which shows the density of the zeros of the partition function with Im $(t) < 0.3$ for gaskets from $\mathcal{G}_{10}$ to $\mathcal{G}_{20}$; in the second case one obtains the result shown in figures 7 and 8 (which refers to $\mathcal{G}_{18}$). From figure 6 one can see that the density does not change appreciably going from one stage to the next except for the tail towards zero that grows longer but is strongly decreasing: the density at $T = 0$ appears to vanish exponentially.

This behaviour can also be verified by an analytical estimate. First we notice that the zeros near $T = 0$ are those obtained by the repeated application of h_2. Indeed for $|x| \to \infty$ we have

$$h_1(x) \to k \tag{24}$$

while

$$h_2(x) \to x + 4 - \frac{16}{x} + O(x^{-2}) \tag{25}$$

and, in terms of the real and imaginary part,

$$h_2(u + iv) \approx \left(u + 4 - \frac{16u}{u^2 + v^2} \right) + i \left(v - \frac{16v}{u^2 + v^2} \right). \tag{26}$$

Applying h_2 to $z = u + iv$ for large u one obtains $h_2(z) \simeq 4u + iv$. In this limit the density of zeros in the x plane becomes the product of two factors, one depending only on u and the other on v:

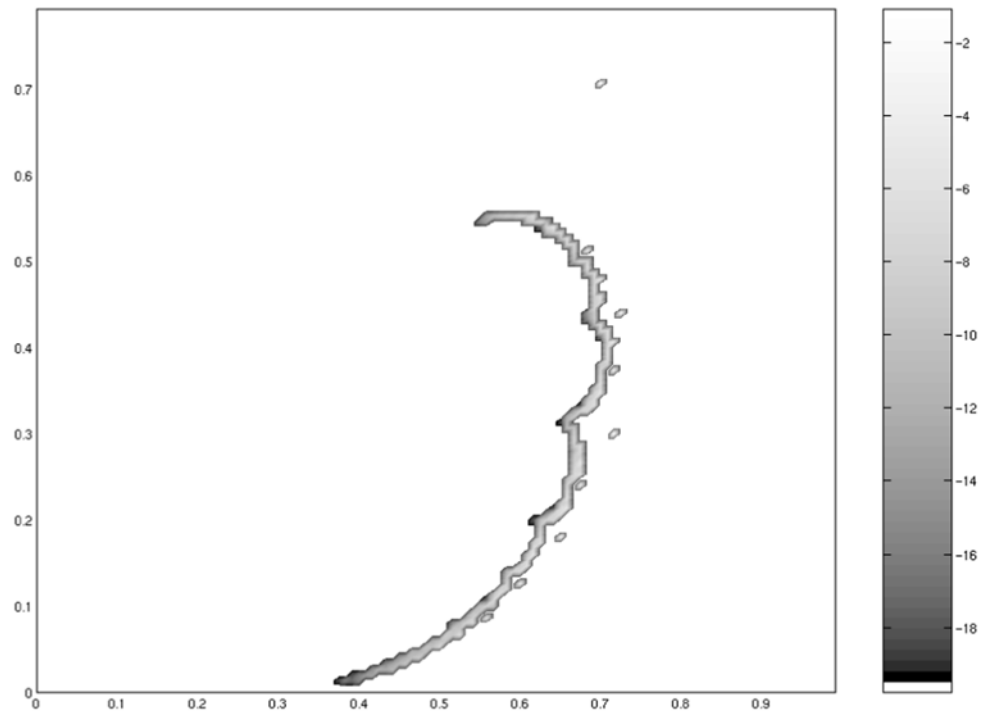$$d(u, v) \approx d_1(u)d_2(v) \tag{27}$$

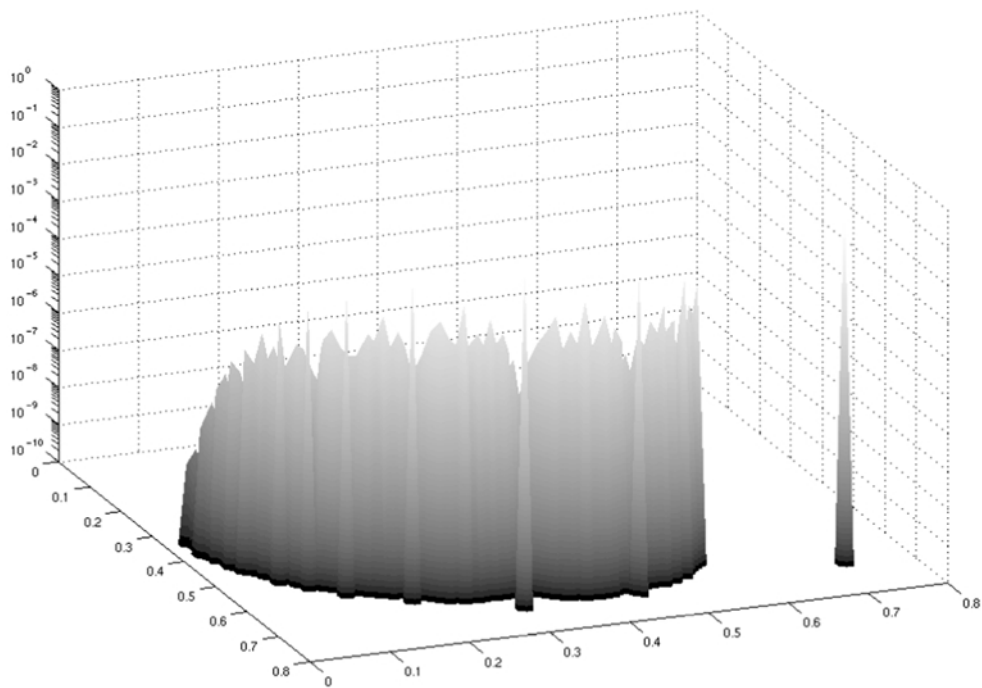**Figure 7.** Density of zeros in the *t* plane (log scale).

Figure 8. Three-dimensional view of the zeros' density in the *t* plane.
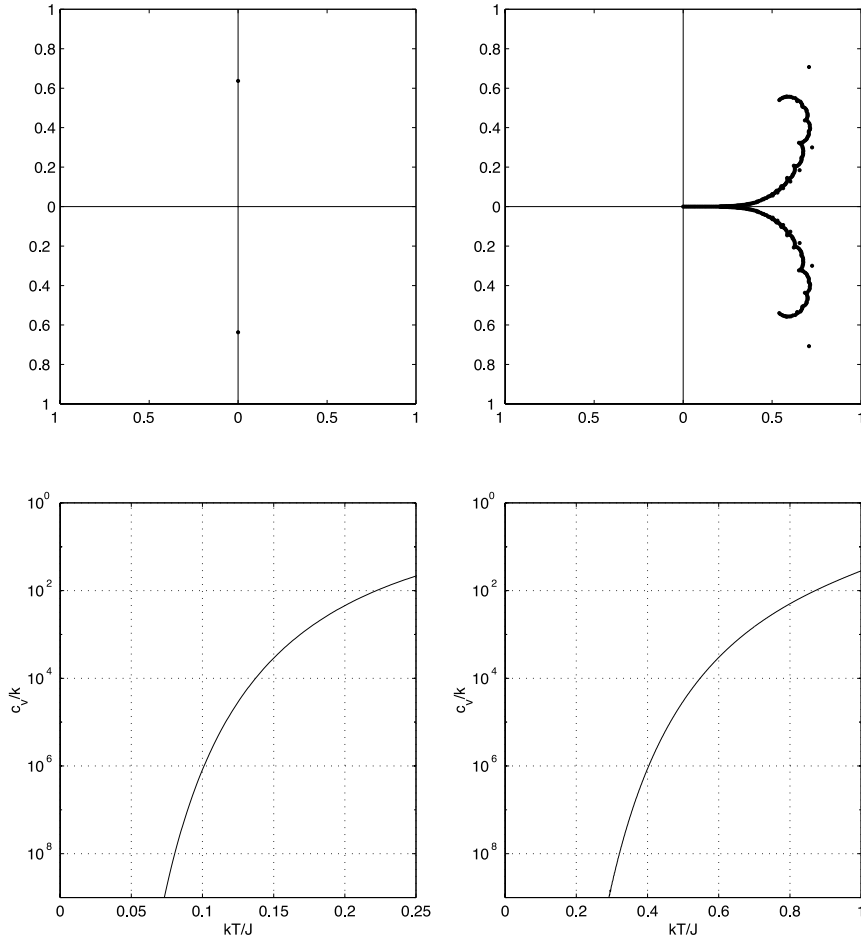
Figure 9. Comparison between the zeros' distribution in the t plane and the specific heat for the linear chain (left) and the gasket (right).

where $d_2(v)$ is bounded.

The asymptotic behaviour of $d_1(u)$ for $u \to \infty$ can be obtained by noting that, for each set $h^k(-1)$, the zeros with real part u are those obtained by a sequence ending with the application of h_1 followed by h_2^n, where $n = \frac{u}{4} + c$ and c is a constant independent of u. So this fraction of zeros is $\frac{1}{2}$ to the power $\frac{u}{4} + c$, that is proportional to $\exp(-u/U)$ with $U = 4 \ln 2$. Since the total density d_1 is a weighted sum of the partial densities that have the same behaviour we have

$$d_1(u) \propto \exp(-u/U). \tag{28}$$

To find the density in the t plane we must now divide d by the Jacobian of the transformation:

$$t = x^{-\frac{1}{4}}. \tag{29}$$

This Jacobian turns out to be

$$\frac{1}{16}(u^2 + v^2)^{-\frac{5}{4}} \tag{30}$$

and finally

$$\tilde{d}(t_r, t_i) \propto d_2(v) e^{-u/U} (u^2 + v^2)^{\frac{5}{4}}|_{t_r, t_i} \tag{31}$$

where t_r, e, t_i are the real and imaginary parts of t.

For $t \to 0$ (that is $u \to +\infty$) the density vanishes exponentially, as we could infer from numerical calculation, and this behaviour has the same effect as a gap near the real axis. Therefore, the low-temperature regime is not affected by the zeros contained in this region and one observes a situation analogous to the one-dimensional case. This can be seen, for example, by comparing the behaviour of thermodynamical quantities: figure 9 shows that, even if the zeros' distributions seem to be quite different, the behaviour of the specific heat for the Sierpinski gasket and the linear chain is essentially the same.

6. Conclusions

The anomalous behaviour of the density of zeros for the Ising model on the Sierpinski gasket is deeply related to its self-similar geometry. This strongly suggests a careful approach to the analysis of scaling of zeros' density on fractals. In particular, a stimulating open problem is the relation of this scaling with the geometry of a generic self-similar structure and with known anomalous dimensions. An important step in this direction would be the study of Fisher zeros on the more complex case of a Sierpinski carpet, where an exact solution is still lacking but the Ising model is expected to have a phase transition at finite temperature.

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