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Cutting-decimation renormalization for diffusive and vibrational dynamics on fractals

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Abstract

Recently, we pointed out that on a class on non exactly decimable fractals two different parameters are required to describe diffusive and vibrational dynamics. This phenomenon we call *dynamical dimension splitting* is related to the lack of exact decimation invariance for these structures, which turn out to be invariant under a more complex cutting-decimation transform. In this paper we study in detail the dynamical dimension splitting on these fractals analyzing the mathematical properties of the cutting-decimation transform. Our results clarify how the splitting arises from the cutting transform and show that the dynamical dimension degeneracy is a very peculiar consequence of exact decimability.

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1. Introduction

Fractal dynamics is a field of primary importance in the study of physical phenomena in real non-crystalline systems. The analytical study of dynamical properties of fractals requires the introduction of simplified model structures, which are supposed to have properties similar to the real ones. However, as a matter of fact, analytical results are usually restricted to exactly decimable fractals, where one can apply the powerful techniques of exact renormalization group. Exactly decimable fractals, which for example include deterministic finitely ramified fractals, are indeed very peculiar

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structures, characterized by strong restrictions on their topology, which are far from being general and representative of all fractals. Nevertheless, due to the availability of analytical results, their behavior was usually considered as typical. This is the case for the spectral dimension of a fractal. Such a parameter, the spectral dimension $\tilde{d}$, was defined by the asymptotic law [1]:

$$\rho(\omega) \sim \omega^{\tilde{d}-1}\,, \tag{1}$$

where $\rho(\omega)$ is the density of harmonic vibrational modes with frequency ω, or by

$$P_{ii}(t) \sim t^{-\frac{\tilde{d}}{2}}\,, \tag{2}$$

where $P_{ii}(t)$ is the probability of returning to the starting site i after t steps ($t \to \infty$) for a random walker and the exponent $\tilde{d}/2$ is independent of the starting point [2]. In the original definition, Eqs. (1) and (2) were supposed to be equivalent by scaling arguments and in all calculations made on exactly decimable fractals they have been found to be always equal. However, it has been shown [3] that from a mathematical point of view it is not possible to conclude that the two asymptotic behaviors of Eqs. (1) and (2) are equivalent. Therefore, one should distinguish between the *diffusive spectral dimension* defined by Eq. (2) and the *vibrational spectral dimension* defined by Eq. (1), the former being relevant for local quantities and the latter for bulk or average quantities. In the following we will show that the coincidence of the two spectral dimensions is typical of exactly decimable fractals while more general structures present dynamical dimension splitting [4]. Indeed for exactly decimable fractals $\rho(\omega)$ and $P_{ii}(t)$ have the same transformation properties under renormalization and this leads to the so called *spectral dimension degeneracy*. Here we will consider the case of a class of non exactly decimable fractals (called in literature Nice Trees of dimension D, NT_D, [5]) showing that Eqs. (1) and (2) can be calculated applying a generalized renormalization transformation, different from the usual decimation and leaving the structure invariant. This transformation consists of the product of a cutting transformation and a decimation. The key point is that here $\rho(\omega)$ and $P_{ii}(t)$ transform according to different laws under this new renormalization procedure. This leads to independent asymptotic behavior for these quantities and requires the definition of two distinct spectral dimensions we will call $\tilde{d}_D$ and $\tilde{d}_V$, D staying for *diffusive* and V for *vibrational*.

2. Harmonic oscillations and random walks on generic networks

The harmonic oscillations of a generic network of masses m connected by springs of elastic constant K are described by the equations of motion for the displacements x_i of each mass from its equilibrium position:

$$m\frac{d^2}{dt^2}x_i = -K\sum_{j\sim i}(x_i - x_j)\,, \tag{3}$$

where the sum runs over the nearest neighbors of point i.

Eq. (3) can be Fourier transformed with respect to the time giving

$$-\frac{\omega^2}{\omega_0^2}\tilde{x}_i = \sum_{j\sim i}(\tilde{x}_j - \tilde{x}_i)\,, \tag{4}$$

where $\omega_0^2 \equiv K/m$.

The equations describing random walks and harmonic oscillations on a network are formally similar. Let us consider a discrete time random walk (the so-called *blind ant* problem) on a network; the master equation for the probability of being at site i after t steps for a random walker starting from an origin site O at time 0, is

$$P_{Oi}(t+1) - P_{Oi}(t) = 1/z_{\max}\sum_{j\sim i}(P_{Oj}(t) - P_{Oi}(t))\,,$$

$$P_{Oi}(0) = \delta_{iO}\,, \tag{5}$$

where $z_{\max}$ is the maximum coordination number of the network (i.e. the maximum number of nearest neighbors of a point), that is assumed to be finite. Applying the discrete Laplace transform with respect to the time defined by

$$\tilde{P}_{ij}(\varepsilon) = \sum_{t=0}^{\infty}(1+\varepsilon)^{-t}P_{ij}(t) \tag{6}$$

and taking $\varepsilon \to 0$, the system (5) can be written in a form similar to Eq. (4):

$$z_{\max}\varepsilon\tilde{P}_{Oi}(\varepsilon) = \sum_{j\sim i}(\tilde{P}_{Oj}(\varepsilon) - \tilde{P}_{Oi}(\varepsilon)) + \delta_{Oi}\,. \tag{7}$$

After the substitutions $-z_{\max}\varepsilon \to \omega^2/\omega_0^2 = \gamma$ and $\tilde{P}_{Oi}(\varepsilon) \to \tilde{x}_i$, Eqs. (4) and (7) look very similar. However they present a fundamental difference consisting in the term δ_{Oi}, arising from the initial condition in Eq. (5). In addition, the system (4) is homogeneous and defines an eigenvalue problem which has infinite solutions (all the normal modes of the graph). The density of eigenvalues for $\omega \to 0$ depends on $\tilde{d}_V$ and it is not a direct solution of the system, but has to be separately calculated after solving the whole system. On the other hand, the system (7) is inhomogeneous and corresponds to a Cauchy problem, which has only one solution. The behavior of such a solution for $\varepsilon \to 0$ depends on $\tilde{d}_D$ and, in principle, has no direct relation with the density of vibrational modes and with $\tilde{d}_V$. A relation between the random walk probabilities $P_{ii}(t)$ and $\tilde{d}_V$ does indeed exist but it involves the average of the $P_{ii}(t)$ over all points of the graph [3]:

$$\overline{P}(t) = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}P_{ii}(t) \sim t^{-\tilde{d}_V/2} \quad \text{for } t \to \infty \tag{8}$$

Relation (8) does not imply that $\tilde{d}_D = \tilde{d}_V$ since the average $\overline{P}(t)$ has in general a different asymptotic behavior from each $P_{ii}(t)$.

3. Random walks and harmonic oscillations on exactly decimable fractals

Exactly decimable fractals are a restricted class of self-similar structures (i.e. not all self-similar structures are exactly decimable) which are geometrically invariant under site decimation. Examples of exactly decimable fractals are the Sierpinski Gasket [6–9], the T-fractal [10,11], the branched Koch curves [12] and so on (Fig. 1). In general, all deterministic finitely ramified fractals are exactly decimable. The solution of both the random walks and the harmonic oscillations problems can be obtained by standard renormalization group calculations based on a real space decimation procedure [13] as well as on the study of DC resistivity [14].

A geometrical structure is decimation invariant if it is possible to eliminate a subset of points (and all the bonds connecting these points) obtaining a network with the same geometry of the starting one. From a mathematical point of view this corresponds to the possibility of eliminating by substitution a set of equations from system (4) or (7) obtaining a system which is similar to the initial one after a suitable redefinition of the coupling parameter γ. If we consider for example a T-fractal the decimation procedure consists in transforming each "T" made of three bonds in a simple bond connecting two points. As it can be easily verified this operation does not change the geometry of the network but requires a redefinition of the coupling parameter γ. In general, after a decimation step, γ splits into a finite number of different couplings γ_μ, $\mu = 1,\ldots,n$ on geometrically unequivalent points. In the case of the T-fractal there are two kinds of inequivalent points: points having one nearest neighbor and points with three nearest neighbors. This suggests to distinguish between a coupling γ_1 and a coupling γ_3 to be used in equations (4) or (7) where point i has respectively one or three nearest neighbors. Before the decimation we have obviously $\gamma_1 = \gamma_3 = \gamma$ but this distinction is useful to put into evidence the forthcoming splitting.

The splitting of γ in an at most finite number of couplings is a necessary condition for exact decimability. If this condition is fulfilled, the decimation transform can be iterated and linearized near the fixed point $\gamma_\mu = 0$. After the linearization the transformation laws for the T-fractal are

$$\gamma_1 \to \gamma_1' = 3\gamma_1 + 1\gamma_3, \qquad \gamma_3 \to \gamma_3' = 3\gamma_1 + 5\gamma_3 \,. \tag{9}$$

The linearized decimation transform can be represented by a matrix D acting on a vector with components $\gamma_\mu, \mu = 1,\ldots,n$ so that

$$\gamma_\mu' = \sum_\nu D_{\mu\nu}\gamma_\nu \,. \tag{10}$$

In our particular case:

$$D = \begin{pmatrix} 3 & 1 \\ 3 & 5 \end{pmatrix}. \tag{11}$$

Decomposing γ_μ on the basis of eigenvectors of D, as the number of decimation steps goes to ∞, γ_μ' tends to the eigenvector corresponding to the largest eigenvalue of D.

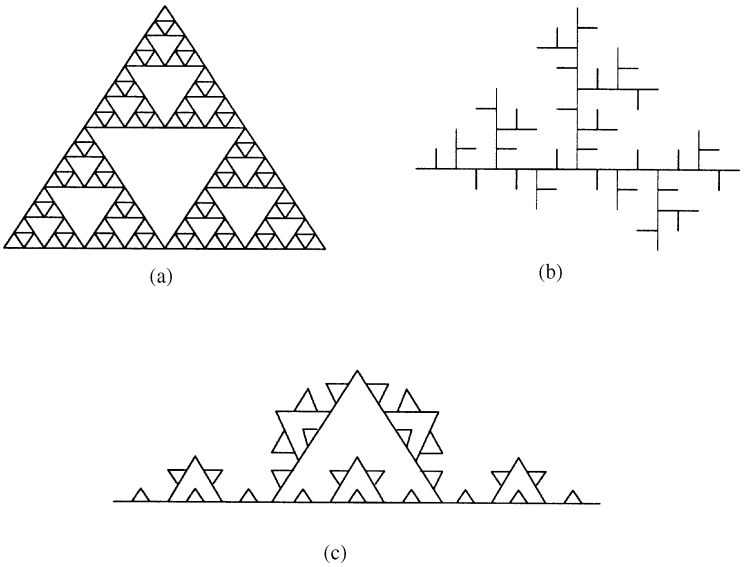

Fig. 1. Exactly decimable fractals: (a) Sierpinski Gasket, (b) T-fractal, (c) branched Koch curves.

Therefore the largest eigenvalue of D, a^2, determines the transformation laws of coupling parameters.

Following these steps in the case of random walks one finds for the parameter ε the transformation:

$$\varepsilon \to \varepsilon'(\varepsilon) \sim a^2 \varepsilon \,, \tag{12}$$

where ε is now the projection of the vector ε_μ on the largest eigenvalue direction. The presence of the term δ_{iO} in Eq. (7) requires a redefinition of the quantities $\tilde{P}_{ij}(\varepsilon)$ to assure that, even after the decimation, the initial condition will correspond to the probability of being in a fixed site equal to 1. One introduces a new parameter c and writes the transformation law for $\tilde{P}_{ij}(\varepsilon)$ as

$$\tilde{P}_{ij}(\varepsilon) \to \tilde{P}'_{ij}(\varepsilon') \sim \frac{1}{c}\tilde{P}_{ij}(\varepsilon) \,. \tag{13}$$

The diffusive spectral dimension $\tilde{d}_D$ is obtained using the relation:

$$\tilde{P}_{OO}(\varepsilon) \sim \varepsilon^{\tilde{d}_D/2-1} \tag{14}$$

which holds only for $\tilde{d}_D < 2$. As we will discuss later, this is always the case for exactly decimable fractals. Now one can rewrite Eq. (13) as

$$\tilde{P}_{ij}(\varepsilon) \sim c\tilde{P}'_{ij}(\varepsilon') \tag{15}$$

and observe that since our graphs are infinite $\tilde{P}_{OO}(\varepsilon)$ and $\tilde{P}'_{OO}(\varepsilon')$ refer to the same structure and they must have the same functional form. This gives

$$\varepsilon^{\tilde{d}_D/2-1} = c(a^2\varepsilon')^{\tilde{d}_D/2-1} \tag{16}$$

so that

$$\tilde{d}_D = 2\frac{\log a^2/c}{\log a^2} \,. \tag{17}$$

As for harmonic oscillations, the invariance of the network under the decimation procedure gives for ω^2 the same transformation law obtained for ε. The analogue of Eq. (12) is here

$$\omega \to \omega'(\omega) \sim a\omega \,. \tag{18}$$

The relation between the density of vibrational modes $\rho(\omega)$ and the new $\rho'(\omega')$ is given by:

$$\rho(\omega)\, d\omega = \rho'(\omega')\, d\omega' \,. \tag{19}$$

Since in the decimation procedure the initial set of Eq. (4) is reduced of a factor r, condition (19) leads to the relation:

$$\rho(\omega) \to \rho'(\omega') \sim \frac{r}{a}\rho(\omega) \tag{20}$$

From Eq. (20) it follows:

$$\rho(\omega) = \frac{a}{r}\rho'(\omega') \tag{21}$$

so that

$$\omega^{\tilde{d}_V - 1} = \frac{a}{r}(a\omega)^{\tilde{d}_V - 1} \,, \tag{22}$$

and

$$\tilde{d}_V = \frac{\log r}{\log a} \,. \tag{23}$$

Comparing Eqs. (17) and (23) one realizes that $\tilde{d}_V = \tilde{d}_D$ if the decimation ratio r is given by

$$r = a^2/c \,. \tag{24}$$

This can be shown to be the case for exactly decimable fractals, using results obtained [2] for the Gaussian model on exactly decimable fractals. The Gaussian model on a graph is defined by the Hamiltonian:

$$H(\{m_i\}) = \frac{J}{4}\sum_{i\sim j}(\phi_i - \phi_j)^2 + \sum_i m_i^2\phi_i^2 \,. \tag{25}$$

The autocorrelation functions of the model are related to the generating functions $\tilde{P}_{ii}(\varepsilon)$ of random walks by

$$\tilde{P}_{ii}(\varepsilon) = \langle \phi_i \phi_i \rangle_{\{m_i^2\}} \frac{z_i}{1-\varepsilon} \,, \tag{26}$$

where z_i is the coordination number of site i and the masses m_i^2 are given by $m_i^2 = z_i\varepsilon/(1-\varepsilon)$. Now it can be shown that a decimation procedure implies the following scaling relations for the coupling J and the masses m_i:

$$J \to \alpha J, \quad m_i \to \beta m_i \,, \tag{27}$$

where the parameters α and β are related to the expression of the diffusive spectral dimension by

$$\tilde{d}_D = \frac{2\log\beta}{\log\beta/\alpha}\,. \tag{28}$$

Now the scaling relations (27) can be rewritten as scaling relations for the fields ϕ_i and the masses m_i:

$$\phi_i \to \sqrt{\alpha}\phi_i, \qquad m_i \to \frac{\beta}{\alpha} m_i\,. \tag{29}$$

In terms of random walks the transformation on the masses in Eq. (29) does not affect the asymptotic behavior of $\tilde{P}_{ii}(\varepsilon)$ [3] while the first of Eq. (29), thanks to relation (26), implies that $\tilde{P}_{ii}(\varepsilon) \to \alpha\tilde{P}_{ii}(\varepsilon)$ so that we can identify α and $1/c$. Since the parameter β is the decimation ratio r, from Eqs. (28), (23) and (17) equality (24) follows.

Notice also that since it is always $\beta > \alpha \neq 1$ [2], exactly decimable fractals have $\tilde{d}_D = \tilde{d}_V < 2$. From Eq. (24), the dimensional degeneracy $\tilde{d}_D = \tilde{d}_V$ can be obtained knowing only two out of the three parameters a, r and c. For example the Sierpinski Gasket has $r = 3$ and $a = \sqrt{5}$, the T-fractal has $r = 3$ and $a = \sqrt{6}$.

4. Random walks and harmonic oscillations on non exactly decimable fractals: NT_D and the cutting-decimation renormalization transform

In the previous section we have analyzed spectral dimension degeneracy on exactly decimable fractals. Now we will consider the case of a class of fractal graphs (NT_D) which are invariant under a more complex transformation $T = D \cdot C$ consisting of the product of a cutting transform C and a decimation D. We call T a cutting-decimation transform. It will be shown that $\tilde{P}_{ii}(\varepsilon)$ and $\rho(\omega)$ behave differently under T and that NT_D are an explicit example of fractals with different diffusive and vibrational spectral dimensions.

The fractal trees known as NT_D [5] can be recursively defined as follows: an origin point O (Fig. 2) is connected to a point 1 by a link, of unitary length; from 1, the tree splits in k branches of length 2 (i.e. consisting of two consecutive links); the ends of these branches split in k branches of length 4 and so on; each endpoint of a branch of length 2^n splits in k branches of length 2^{n+1}. NT_D can be naturally embedded in a suitable Euclidean space in such a way that their fractal dimension d_F coincides with their connectivity dimension $d_C = 1 + \ln k/\ln 2$ [15].

As one can easily verify, NT_D are not exactly decimable since after a simple decimation starting from the origin O, one obtains k copies of the original structure joined together in a point instead of the same NT_D. The NT_D invariance under a T transform can be described in the following way (Fig. 3). Suppose to cut the log of the tree in point 1 and to separate the k branches (cutting transform). Now, each branch can be obtained from the initial NT_D by a dilatation with a factor 2. Eliminating all branches but one and decimating it (decimation transform), one obtains the original NT_D. The

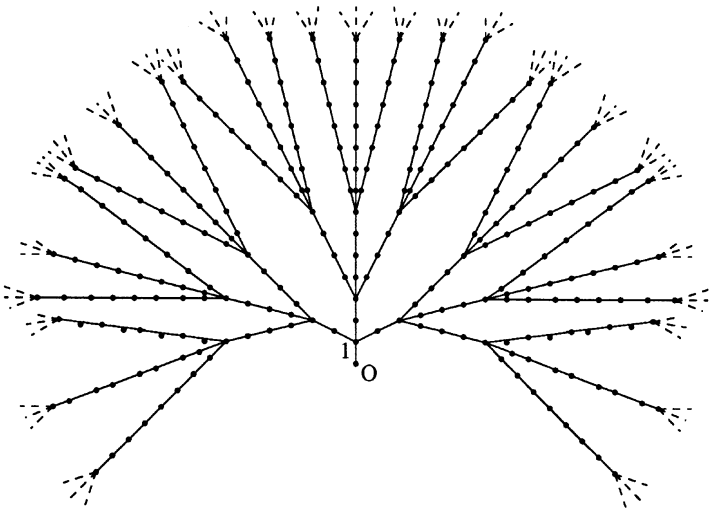

Fig. 2. An example of non-exactly decimable fractal: an NT_D with $k = 3$.

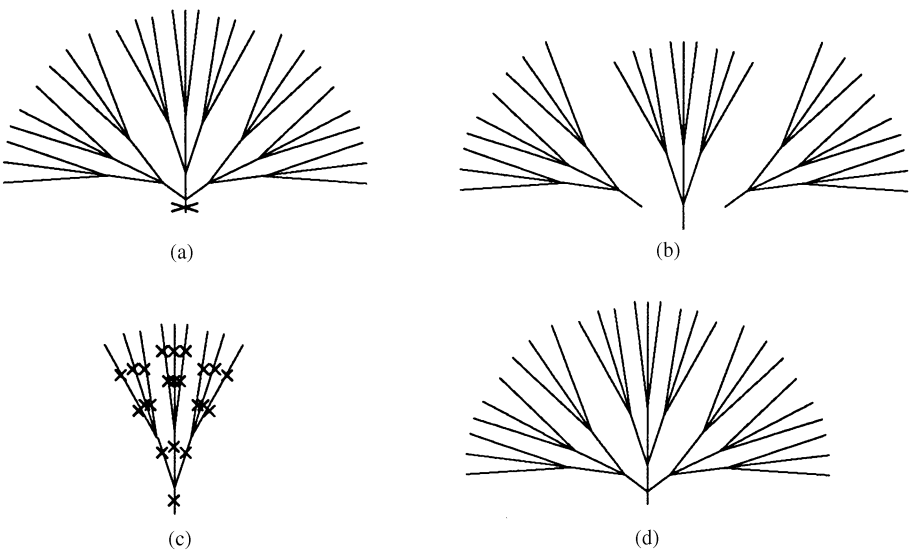

Fig. 3. Cutting-decimation procedure: (a) cutting of the log of the NT_D, (b) separation of the k branches, (c) decimation of the points labelled by X, (d) recovering of the original NT_D.

T transform can be applied to solve the random walks problem. The cutting transform gives a relation between random walks on the whole tree and random walks on one of its branches; more precisely one relates $\tilde{P}_{OO}^{\text{tree}}(\varepsilon)$, the generating function of the probability of returning to point O after a random walk on the NT_D tree, and $\tilde{P}_{11}^{\text{branch}}(\varepsilon)$, the generating function of the probability of returning to the starting point 1 after a

random walk on one of the branches. This relation is given by [15]

$$\tilde{P}_{OO}^{\text{tree}}(\varepsilon) = \frac{\tilde{P}_{11}^{\text{branch}}(\varepsilon) + k}{2\varepsilon\tilde{P}_{11}^{\text{branch}}(\varepsilon) + k} \,. \tag{30}$$

The decimation transform on the surviving branch is obtained as for exactly decimable fractals by the transformation laws

$$\varepsilon \to a^2\varepsilon = 4\varepsilon \,,$$

$$\tilde{P}_{11}^{\text{branch}}(\varepsilon) \to \widetilde{P}'^{\text{branch}}_{11}(\varepsilon') = \frac{1}{c}\tilde{P}_{11}^{\text{branch}}(\varepsilon) = \frac{1}{2}\tilde{P}_{11}^{\text{branch}}(\varepsilon) \,. \tag{31}$$

Now since the decimation procedure transforms a branch into the initial tree we have $\widetilde{P}'^{\text{branch}}_{11}(\varepsilon') = \tilde{P}_{OO}^{\text{tree}}(\varepsilon')$ and relation (30) becomes:

$$\tilde{P}_{OO}^{\text{tree}}(\varepsilon) = \frac{2\tilde{P}_{OO}^{\text{tree}}(4\varepsilon) + k}{4\varepsilon\tilde{P}_{OO}^{\text{tree}}(4\varepsilon) + k} \,. \tag{32}$$

Choosing a suitable expression for $P_{OO}^{\text{tree}}(\varepsilon)$ [15] we obtain from (32):

$$\tilde{d}_D = 1 + \frac{\log k}{\log 2} \,. \tag{33}$$

As for harmonic oscillations, the cutting transform gives a relation between $\rho^{\text{tree}}(\omega)$ and $\rho^{\text{branch}}(\omega)$; this can be obtained using the following properties [16]: (1) cutting or adding a finite number of points or bonds from an infinite network does not affect its vibrational spectrum; (2) the spectral density $\rho(\omega)$ (normalized to 1) of k copies of a given structure all attached in a point coincides with the corresponding one for a single structure. From these properties it immediately follows that the cutting transform does not affect the density of modes $\rho(\omega)$ so that

$$\rho^{\text{tree}}(\omega) = \rho^{\text{branch}}(\omega) \,. \tag{34}$$

Applying the decimation transform we get $a = 2$ from Eq. (31) while the decimation ratio is $r = 2$ so that (20) becomes

$$\rho^{\text{branch}}(\omega) \to \rho'^{\text{branch}}(\omega') \sim \rho^{\text{branch}}(\omega) \,. \tag{35}$$

Since $\rho^{\text{branch}}(\omega') = \rho^{\text{tree}}(\omega')$ relations (34) and (35) give us $\rho^{\text{tree}}(\omega) = \rho^{\text{tree}}(\omega')$ so that $\tilde{d}_V = 1$.

Since $\tilde{d}_D = 1 + \log k/\log 2$ and $\tilde{d}_V = 1$, NT_D trees represent an explicit case of dynamical dimension splitting. All our results can be generalized [17] to NT_D lattices with a branches growth factor r (or decimation ratio) different from 2. In all these cases we will find again dynamical dimension splitting since $\tilde{d}_V = 1$ and

$$\tilde{d}_D = \tilde{d}_V + \frac{\log k}{\log r} \,. \tag{36}$$

5. Conclusions

In this paper we have shown how to build an exact renormalization group technique in real space using a more general reduction procedure than the usual decimation

transform. This new technique allows to point out the phenomenon of dynamical dimension splitting which, as we have explicitly shown, is always absent in the case of exactly decimable fractals, where $\tilde{d}_D = \tilde{d}_V < 2$. Therefore these structures have very peculiar statistical mechanical properties and their study does not allow to distinguish between average and local quantities.

Many examples of spectral dimension splitting can be found even on non fractal-graphs, such as comb lattices [18] and other branched structures [19]. In general the situation with $\tilde{d}_D \neq \tilde{d}_V$ is quite common so that it becomes particularly important to study the consequences of dynamical dimension splitting in all the phenomena which are influenced by the value of the spectral dimension such as diffusion, vibrational dynamics and phase transitions.

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